

Algebraic structures on walks of graphs and algebraic reconstruction of the identity

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Outlines

- 1 Combinatorial Hopf algebras
 - Definition of a combinatorial Hopf algebra
 - Tensor Hopf algebra
- 2 Hopf algebras on walks on graphs
 - Lawler's procedure
 - copre-Lie coalgebra
 - Tensor and symmetric Hopf algebras
 - Antipode
- 3 Reconstruction of the identity
 - Obstacle
 - Reconstruction of the identity
 - Example

Specificities of a Hopf algebra

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 \downarrow Id \otimes m & & \downarrow m \\
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 \mathbb{K} \otimes \mathcal{H} & \xrightarrow{\eta \otimes Id} & \mathcal{H} \otimes \mathcal{H} & \xleftarrow{Id \otimes \eta} & \mathcal{H} \otimes \mathbb{K} \\
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- $\eta : \begin{cases} \mathbb{K} & \longrightarrow \mathcal{H} \\ k & \longrightarrow k.1_{\mathcal{H}} \end{cases}$

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- $\varepsilon(k.1_{\mathcal{H}}) = k$ and $\varepsilon(h) = 0$ if $h \notin H_0$.

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- The co-unit ε is the unit of the product \star .
- The bialgebra $(\mathcal{H}, m, \eta, \Delta, \varepsilon)$ is a Hopf algebra if there exists $S \in \mathcal{H}^{\otimes}$ such that

$$S \star \text{Id} = \text{Id} \star S = \varepsilon.$$

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- Δ and ε morphisms of algebras.
- The antipode S exists.

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- Let V be a graded vector space such that $V_0 = (0)$. The Tensor space $T\langle V \rangle$ is defined by:

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$$V_1 \dots V_n \cdot W_1 \dots W_s = V_1 \dots V_n W_1 \dots W_s.$$

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$$\begin{aligned} \Delta(v) &= v \otimes 1 + 1 \otimes v \text{ for any } v \in V, \\ \Delta(v_1 v_2) &= \Delta(v_1 \cdot v_2) = \Delta(v_1) \Delta(v_2) \\ &= (v_1 \otimes 1 + 1 \otimes v_1)(v_2 \otimes 1 + 1 \otimes v_2) \\ &= v_1 v_2 \otimes 1 + v_1 \otimes v_2 + v_2 \otimes v_1 + 1 \otimes v_1 v_2. \end{aligned}$$

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- Coassociative counit coproduct: v is a primitive element for any $v \in V$.
- Antipode: $S(v_1 \dots v_n) = (-1)^n v_n \dots v_1$.

Definition

Let Γ be a connected graph, $\omega = w_1 \dots w_m$ be a walk in Γ and $\text{Nod}(\omega)$ the set of the nodes of Γ visited by ω .


- 1 The walk ω is called a self-avoiding walk if $\text{Nod}(\omega)$ is a set of cardinality m .
- 2 The walk ω is called a simple cycle if $w_1 = w_m$ and $\text{Nod}(\omega)$ is a set of cardinality $m - 1$.

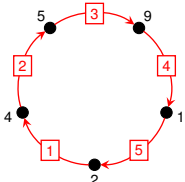
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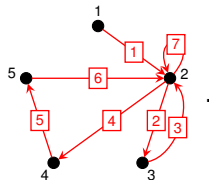
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Examples

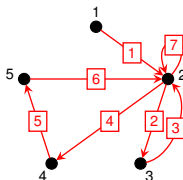
$\rho = 15324 =$  is a self-avoiding walk and

$\mu = 245912 =$  a simple cycle.

Let consider the walk $\sigma = 12324522 =$

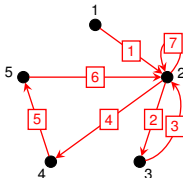


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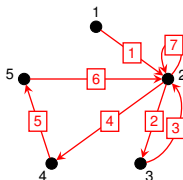


. With the

Lawler's loop erasing procedure we get:

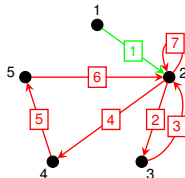


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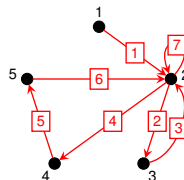


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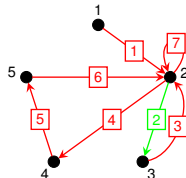


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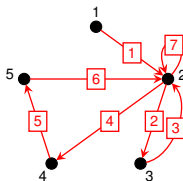


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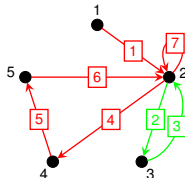


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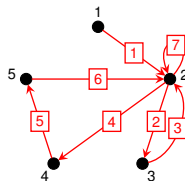


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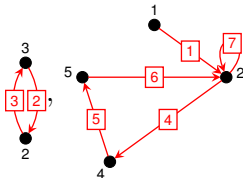


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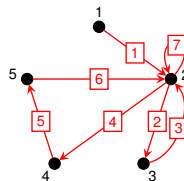


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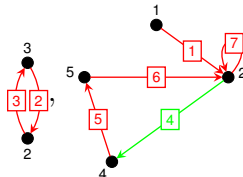


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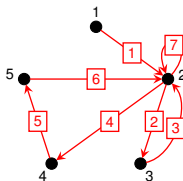


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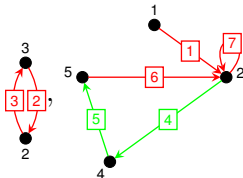


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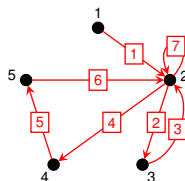


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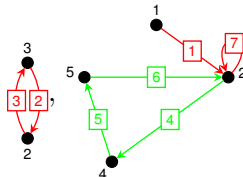


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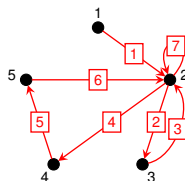


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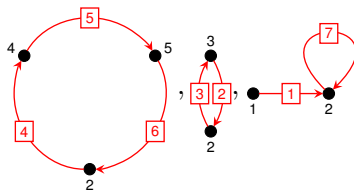


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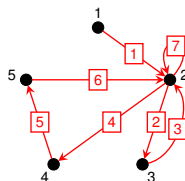


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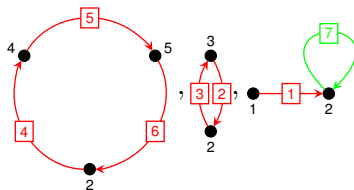


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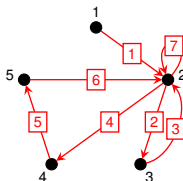


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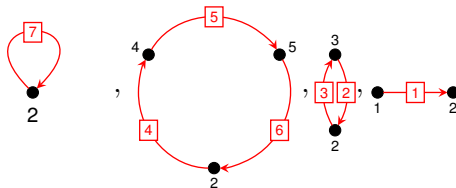


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Let $\omega = w_1 \dots w_m$ be a walk in a finite or countable connected graph Γ . We say that a walk $\omega^{II'} := w_I w_{I+1} \dots w_{I'}$ is an admissible cut of ω when it satisfies all of the following conditions

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- ② $w_l = w_{l'}$, i.e. $\omega^{ll'}$ is a closed walk;
- ③ $\omega^{ll'}$ is a cycle erased by the Lawler's procedure
- ④ Let \mathcal{L} be the set of loop-erased cycles $\omega^{kk'}$ of ω such that $k' > l'$ and $\omega^{ll'}$ is included in $\omega^{kk'}$.

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Let $\omega = w_1 \dots w_m$ be a walk in a finite or countable connected graph Γ . We say that a walk $\omega^{l'l} := w_l w_{l+1} \dots w_{l'}$ is an admissible cut of ω when it satisfies all of the following conditions

- ① $\omega^{l'l} \neq \omega$ and $\omega^{l'l} \neq ()$ where $()$ is the empty walk;
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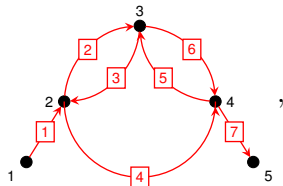
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 - Either $\mathcal{L} = \emptyset$
 - or the minimum element $\omega^{kk'}$ for inclusion satisfies the statement: the letter w_l does not appear in $w_{l'+1} \dots w_{k'}$.

The set of admissible cuts of ω is denoted $AdC(\omega)$.

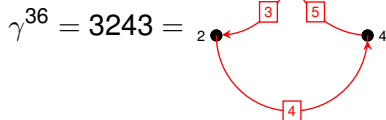
Example

In the walk

$$\gamma = 12324345 =$$



the subwalk



is not an admissible cut.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ be a finite or a countable connected graph and $\mathcal{W}(\Gamma)$ the vector space spanned by its walks. Let define the linear map Δ_{CP} by:

$$\Delta_{CP} : \begin{cases} \mathcal{W}(\Gamma) & \longrightarrow & \mathcal{W}(\Gamma) \otimes \mathcal{W}(\Gamma) \\ \omega & \longmapsto & \Delta_{CP}(\omega) = \sum_{\omega''' \in \text{AdC}(\omega)} \omega_{||'} \otimes \omega''', \end{cases}$$

where $\omega = w_1 \dots w_m$ is a walk, $\omega_{||'} = w_1 \dots w_l w_{l'+1} \dots w_m$ and the sum is taken over all the admissible cuts of ω . Then the vector space $\mathcal{W}(\Gamma)$, equipped with the coproduct Δ_{CP} , is a co-preLie (not counit) coalgebra ie Δ_{CP} satisfies the relation

$$(\Delta_{CP} \otimes \text{Id} - \text{Id} \otimes \Delta_{CP}) \circ \Delta_{CP}(\omega) = (\text{Id} \otimes \tau) \circ (\Delta_{CP} \otimes \text{Id} - \text{Id} \otimes \Delta_{CP}) \circ \Delta_{CP}(\omega)$$

Example

$$\Delta_{\text{CP}} \left(\begin{array}{c} \text{Diagram 1: A cycle } i \rightarrow j \rightarrow k \rightarrow i \text{ with edges labeled } 1, 2, 3, 4, 5. \text{ Edges } 2 \text{ and } 4 \text{ are loops at } j \text{ and } k \text{ respectively.} \end{array} \right) = \begin{array}{c} \text{Diagram 2: Same cycle as Diagram 1, but with edge } 2 \text{ removed.} \\ \text{Diagram 3: Same cycle as Diagram 1, but with edge } 4 \text{ removed.} \end{array} \otimes \begin{array}{c} \text{Diagram 4: Loop at } j \text{ with edge } 2. \\ \text{Diagram 5: Loop at } k \text{ with edge } 4. \end{array} + \begin{array}{c} \text{Diagram 6: Same cycle as Diagram 1, but with edge } 1 \text{ removed.} \end{array} \otimes \begin{array}{c} \text{Diagram 7: Loop at } i \text{ with edge } 1. \end{array}$$

The diagram illustrates the coproduct Δ_{CP} applied to a cycle graph with three vertices i, j, k and five edges labeled 1 through 5. The cycle is $i \rightarrow j \rightarrow k \rightarrow i$. Edges 2 and 4 are loops at vertices j and k respectively. The coproduct is the sum of two terms, each consisting of a tensor product of two graphs. The first term shows the removal of edges 2 and 4, resulting in two graphs: one with edge 2 removed and one with edge 4 removed, each tensored with a loop at the corresponding vertex. The second term shows the removal of edge 1, resulting in a graph with edge 1 removed tensored with a loop at vertex i .

Definition

Let Γ be a finite or a countable connected graph and $\omega = w_1 \dots w_m$ be a walk in Γ . An extended admissible cut of ω is a sequence

$$1 \leq l_1 < l'_1 < l_2 < l'_2 < \dots < l_s < l'_s \leq m$$

satisfying that $\omega^{l_k l'_k}$ is an admissible cut of ω , for any $1 \leq k \leq s$. The set of extended admissible cuts of ω is denoted $EAdC(\omega)$.

Definition

We define the morphism of algebras Δ_H defined by:

$$\Delta_H : \begin{cases} \mathcal{T}\langle \mathcal{W}(\Gamma) \rangle & \longrightarrow & \mathcal{T}\langle \mathcal{W}(\Gamma) \rangle \otimes \mathcal{T}\langle \mathcal{W}(\Gamma) \rangle \\ \omega & \mapsto & \Delta_H(\omega) = \omega \otimes \mathbf{1} + \mathbf{1} \otimes \omega \\ & & + \sum_{c \in EAdC(\omega)} \omega_{l_1 l'_1, \dots, l_s l'_s} \otimes \omega^{l_1 l'_1} | \dots | \omega^{l_s l'_s}, \end{cases}$$

where $\omega = w_1 \dots w_m$ is a walk in Γ , the extended admissible cut c is the sequence $1 \leq l_1 < l'_1 < \dots < l_s < l'_s \leq m$ and the sum is taken over all the extended admissible cuts of ω .

Example

$$\begin{aligned}
 \Delta_H \left(\text{Diagram 1} \right) &= \text{Diagram 1} \otimes 1 \\
 &+ 1 \otimes \text{Diagram 2} \\
 &+ \text{Diagram 3} \otimes \text{Diagram 4} \mid \text{Diagram 5} \\
 &+ \Delta_{CP} \left(\text{Diagram 1} \right).
 \end{aligned}$$

Diagram 1: A directed cycle graph with three vertices labeled i , j , and k . The edges are labeled with red boxes containing numbers: $i \rightarrow j$ is 2, $j \rightarrow k$ is 3, $k \rightarrow i$ is 5, and there is a self-loop at j labeled 4.

Diagram 2: A directed cycle graph with three vertices labeled i , j , and k . The edges are labeled with red boxes containing numbers: $i \rightarrow j$ is 2, $j \rightarrow k$ is 3, $k \rightarrow i$ is 5, and there is a self-loop at k labeled 4.

Diagram 3: A directed cycle graph with three vertices labeled i , j , and k . The edges are labeled with red boxes containing numbers: $i \rightarrow j$ is 2, $j \rightarrow k$ is 3, $k \rightarrow i$ is 5, and there is a self-loop at j labeled 2.

Diagram 4: A directed cycle graph with three vertices labeled i , j , and k . The edges are labeled with red boxes containing numbers: $i \rightarrow j$ is 2, $j \rightarrow k$ is 3, $k \rightarrow i$ is 5, and there is a self-loop at k labeled 4.

Diagram 5: A directed cycle graph with three vertices labeled i , j , and k . The edges are labeled with red boxes containing numbers: $i \rightarrow j$ is 2, $j \rightarrow k$ is 3, $k \rightarrow i$ is 5, and there is a self-loop at j labeled 4.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ a finite or countable connected graph. Consider the triple $(\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle, \star, \Delta_H)$. It is a Hopf algebra.

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Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

In the graph Γ , we denote by \mathcal{I} the vector space spanned by the elements $\omega_1 | \dots | \omega_s - \omega_{\sigma(1)} | \dots | \omega_{\sigma(s)}$ where $\omega_1 | \dots | \omega_s \in \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$ and σ is a permutation. Then, \mathcal{I} is a Hopf bi-ideal of $\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$. Thus, $(\mathcal{S}\langle\mathcal{W}(\Gamma)\rangle, \square, \Delta_H)$ is a quotient Hopf algebra of $\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$.

Definition

Let Γ be a finite or a countable connected graph and $\omega = w_1 \dots w_m$ be a walk in Γ such that $AdC(\omega) \neq \emptyset$. For any $(\omega^{kk'}, \omega^{ll'})$ in $AdC(\omega)^2$, the two following statements are equivalent:

- ① $\omega^{kk'} \leq \omega^{ll'}$.
- ② $l \leq k < k' \leq l'$ or $k < k' < l < l'$.

Definition

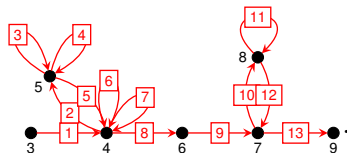
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Example

Let consider the walk

$$\psi = 34555444678879 =$$



Let $\psi^{35} = 555$, $\psi^{45} = 55$ and $\psi^{11\ 12} = 88$ be three elements in $AdC(\psi)$. Then, $\psi^{45} \leq \psi^{35} \leq \psi^{11\ 12}$.

Proposition (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let ω be a non-empty finite walk such that $AdC(\omega) \neq \emptyset$. Equipped with the relation \leq , the set $AdC(\omega)$ is a totally ordered set.

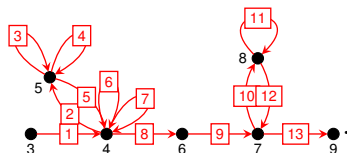
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Example

Consider the walk

$$\psi = 34555444678879 =$$



Then

$$AdC(\psi) = \{\psi^{45} \leq \psi^{35} \leq \psi^{78} \leq \psi^{68} \leq \psi^{28} \leq \psi^{11\ 12} \leq \psi^{10\ 13}\}.$$

Definition

Let ω be a non-empty finite walk, $AdC(\omega)$ be the set of its admissible cuts and $x(\omega) \in \mathbb{N}$ be the cardinality of $AdC(\omega)$. We assume $AdC(\omega) \neq \emptyset$. Let $s \in \{1, \dots, x\}$ be a positive integer and $(\omega^{l_1 l'_1}, \dots, \omega^{l_s l'_s})$ be a s -tuple of distinct admissible cuts of ω such that $\omega^{l_1 l'_1} \leq \dots \leq \omega^{l_s l'_s}$. We associate to this s -tuple a tensor $T_{l_1 l'_1, \dots, l_s l'_s}$ as follows:

$$T_{l_1 l'_1, \dots, l_s l'_s} = \omega_{l_1 l'_1, \dots, l_s l'_s} | \omega_{l_1 l'_1, \dots, l_{s-1} l'_{s-1}}^{l_s l'_s} | \cdots | \omega_{l_1 l'_1, \dots, l_{i-1} l'_{i-1}}^{l_i l'_i} | \cdots | \omega^{l_1 l'_1}.$$

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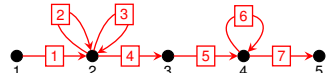
$$T_{l_1 l'_1, \dots, l_s l'_s} = \omega_{l_1 l'_1, \dots, l_s l'_s} | \omega_{l_1 l'_1, \dots, l_{s-1} l'_{s-1}}^{l_s l'_s} | \cdots | \omega_{l_1 l'_1, \dots, l_{i-1} l'_{i-1}}^{l_i l'_i} | \cdots | \omega^{l_1 l'_1}.$$

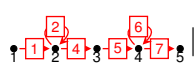
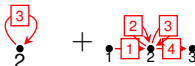






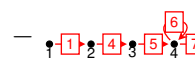

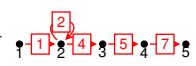

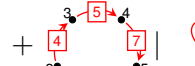



Theorem

Let Γ be a finite connected graph and ω a non-empty finite walk in Γ . Then, in $\mathcal{T}\langle \mathcal{W}(\Gamma) \rangle$, the antipode $S(\omega)$ calculated on ω is given by:

$$S(\omega) = -\omega - \sum_{s=1}^{x(\omega)} \sum_{\substack{\omega^{l_1 l'_1} \leq \dots \leq \omega^{l_s l'_s} \\ \forall i, \omega^{l_i l'_i} \in AdC(\omega) \\ \forall i \neq j, \omega^{l_i l'_i} \neq \omega^{l_j l'_j}}} (-1)^s T_{l_1 l'_1, \dots, l_s l'_s}$$

Example

Consider the walk $\kappa = 12223445 =$ . Then $AdC(\kappa) = \{\kappa^{34} \leq \kappa^{24} \leq \kappa^{67}\}$ and the antipode of κ is

$$S(\kappa) = -\kappa +$$

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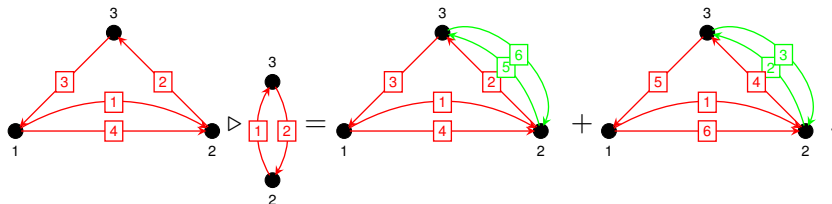
Pre-Lie product

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The pre-Lie product on walks is given by the linear part in the dual product of Δ_H .

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$$12312 \triangleright 232 = 1231232 + 1232312.$$

There are two terms in the result.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let ω walk in a connected finite or countable graph Γ . There exists a unique integer p , a unique p -tuple $(\omega_1, \dots, \omega_p)$ of walks such that:

$$\omega = \text{Parentheses}((\omega_1, \omega_2, \dots, \omega_p), \triangleright).$$

Besides:

- 1 *The reconstruction is essentially unique,*
- 2 *$\omega_2, \dots, \omega_p$ are simple cycles,*
- 3 *ω_1 is a self-avoiding walk or a simple cycle.*

Questions:

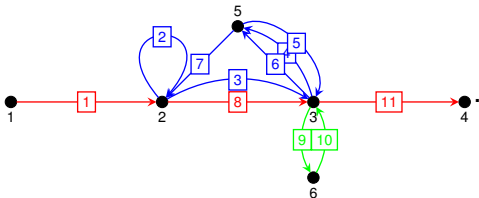
- 1 How find the integer p and the walks $\omega_1, \dots, \omega_p$?
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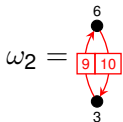
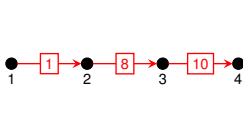
- 1 How find the integer p and the walks $\omega_1, \dots, \omega_p$?
- 2 How find the couples of parentheses?

Answer: You have to use the total order \leq on the set $AdC(\omega)$.

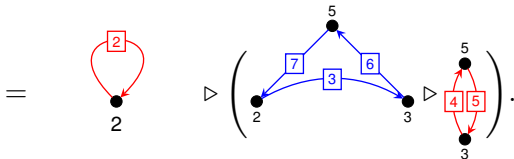
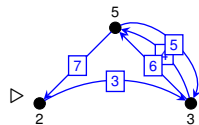
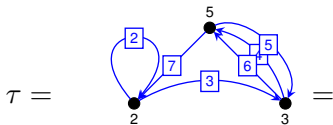
Consider the walk $\omega =$



Then: $\omega_1 =$

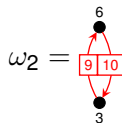


And then,

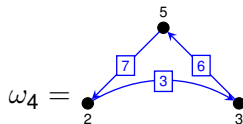


Finally:

$$\omega_1 = \bullet_1 \xrightarrow{1} \bullet_2 \xrightarrow{8} \bullet_3 \xrightarrow{10} \bullet_4,$$



$$\omega_3 = \begin{array}{c} \boxed{2} \\ \uparrow \downarrow \\ \bullet_2 \end{array},$$



$$\omega_5 = \begin{array}{c} \bullet_5 \\ \uparrow \downarrow \\ \boxed{4 \ 5} \\ \uparrow \downarrow \\ \bullet_3 \end{array}.$$

