Algebraic structures on walks of graphs and algebraic reconstruction of the identity

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Outlines

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- $\eta : \begin{cases} \mathbb{K} & \longrightarrow & \mathcal{H} \\ k & \longrightarrow & k.1_{\mathcal{H}} \end{cases}$

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- $\varepsilon(k.1_{\mathcal{H}}) = k$ and $\varepsilon(h) = 0$ if $h \notin H_0$.

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- The bialgebra $(\mathcal{H}, m, \eta, \Delta, \varepsilon)$ is a Hopf algebra if there exists $S \in \mathcal{H}^{\circledast}$ such that

$$S \bigstar \mathrm{Id} = \mathrm{Id} \bigstar S = \varepsilon.$$

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- Δ and ε morphisms of algebras.
- The antipode S exists.

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$$\begin{aligned}
\Delta(v) =& v \otimes 1 + 1 \otimes v \text{ for any } v \in V,\\
\Delta(v_1 v_2) =& \Delta(v_1 \cdot v_2) = \Delta(v_1)\Delta(v_2)\\
=& (v_1 \otimes 1 + 1 \otimes v_1)(v_2 \otimes 1 + 1 \otimes v_2)\\
=& v_1 v_2 \otimes 1 + v_1 \otimes v_2 + v_2 \otimes v_1 + 1 \otimes v_1 v_2.
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- Antipode: $S(v_1 \ldots v_n) = (-1)^n v_n \ldots v_1$.

Definition

Let Γ be a connected graph, $\omega = w_1 \dots w_m$ be a walk in Γ and $\mathrm{Nod}(\omega)$ the set of the nodes of Γ visited by ω.

1. The walk ω is called a self-avoiding walk if $\mathrm{Nod}(\omega)$ is a set of cardinality m.
2. The walk ω is called a simple cycle if $w_1 = w_m$ and $\mathrm{Nod}(\omega)$ is a set of cardinality $m - 1$.

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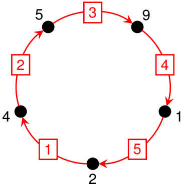
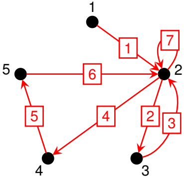
Examples

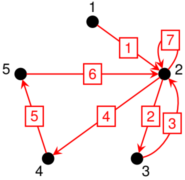
$\rho = 15324 =$ 

is a self-avoiding walk and

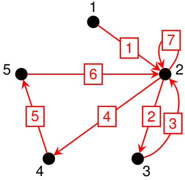
$\mu = 245912 =$ 

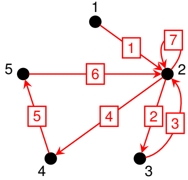
a simple cycle.

Let consider the walk $\sigma = 12324522 =$  .

Let consider the walk $\sigma = 12324522 =$ 

. With the Lawler's loop erasing procedure we get:



Let consider the walk $\sigma = 12324522 =$ 

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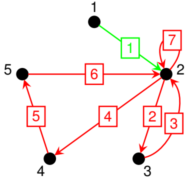
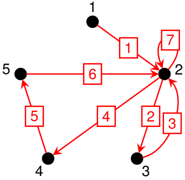


Let consider the walk $\sigma = 12324522 =$ 

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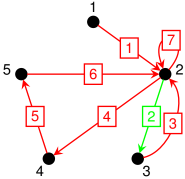
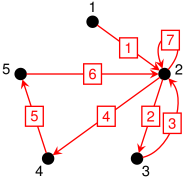


Let consider the walk $\sigma = 12324522 =$ 

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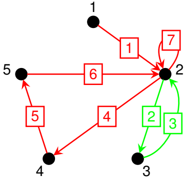
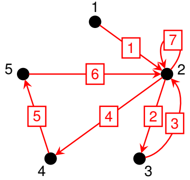
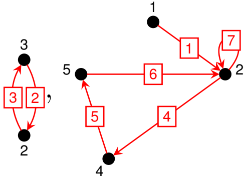
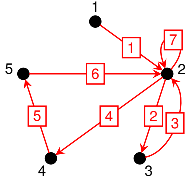


Let consider the walk $\sigma = 12324522 =$  . With the Lawler's loop erasing procedure we get:



Let consider the walk $\sigma = 12324522 =$ 

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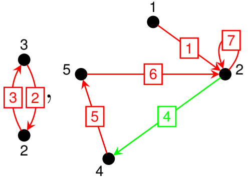
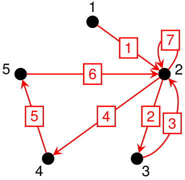


Let consider the walk $\sigma = 12324522 =$  . With the

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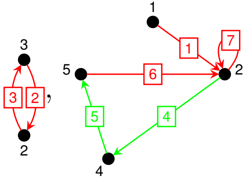
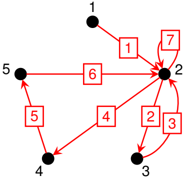
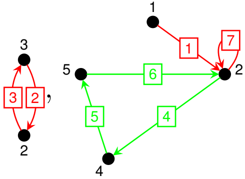
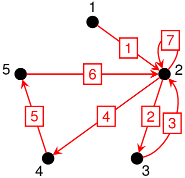


Let consider the walk $\sigma = 12324522 =$  . With the Lawler's loop erasing procedure we get:



Let consider the walk $\sigma = 12324522 =$ 

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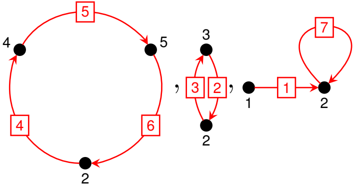
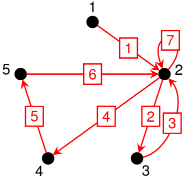


Let consider the walk $\sigma = 12324522 =$  . With the

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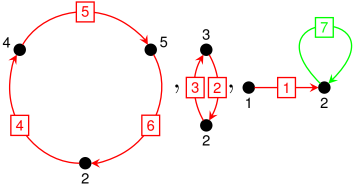
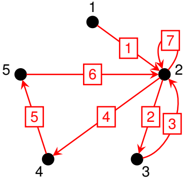
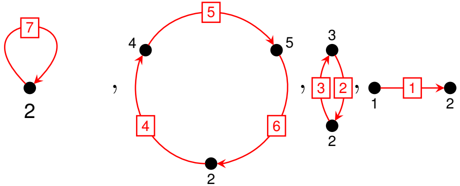


Let consider the walk $\sigma = 12324522 =$ . With the Lawler's loop erasing procedure we get:



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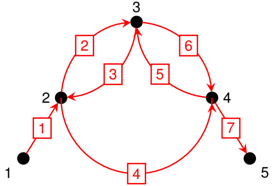
4. Let $\mathcal{L}$ be the set of loop-erased cycles $\omega^{kk'}$ of ω such that $k' > l'$ and $\omega^{ll'}$ is included in $\omega^{kk'}$.
   - Either $\mathcal{L} = \emptyset$
   - or the minimum element $\omega^{kk'}$ for inclusion satisfies the statement: the letter w_l does not appear in $w_{l'+1} \dots w_{k'}$.

The set of admissible cuts of ω is denoted $AdC(\omega)$.

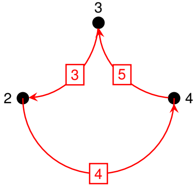
Example

In the walk

$\gamma = 12324345 =$ 

,

the subwalk

$\gamma^{36} = 3243 =$ 

is not an admissible cut.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ be a finite or a countable connected graph and $\mathcal{W}(\Gamma)$ the vector space sanned by its walks. Let define the linear map Δ_{CP} by:

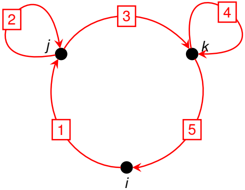
$$\Delta_{CP}:\left\{\begin{array}{ccc} \mathcal{W}(\Gamma) & \longrightarrow & \mathcal{W}(\Gamma)\otimes\mathcal{W}(\Gamma) \\ \omega & \mapsto & \Delta_{CP}(\omega)=\displaystyle\sum_{\omega^{ll'}\in AdC(\omega)} \omega_{ll'}\otimes\omega^{ll'}, \end{array}\right.$$

where $\omega = w_1\dots w_m$ is a walk, $\omega_{ll'} = w_1\dots w_l w_{l'+1}\dots w_m$ and the sum is taken over all the admissible cuts of ω. Then the vector space $\mathcal{W}(\Gamma)$, equipped with the coproduct Δ_{CP}, is a co-preLie (not counit) coalgebra ie Δ_{CP} satisfies the relation

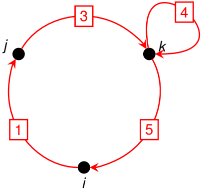
$$(\Delta_{CP}\otimes Id - Id\otimes\Delta_{CP})\circ\Delta_{CP}(\omega) = (Id\otimes\tau)\circ(\Delta_{CP}\otimes Id - Id\otimes\Delta_{CP})\circ\Delta_{CP}(\omega)$$

Example

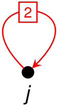
$\Delta_{CP}\Bigg($



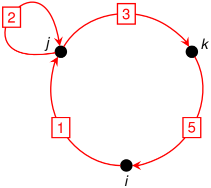
$\Bigg) =$



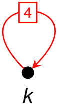
$\otimes$



$+$



$\otimes$



Definition

Let Γ be a finite or a countable connected graph and $\omega = w_1 \dots w_m$ be a walk in Γ. An extended admissible cut of ω is a sequence

$$1 \leq l_1 < l_1' < l_2 < l_2' < \cdots < l_s < l_s' \leq m$$

satisfying that $\omega^{l_k l_k'}$ is an admissible cut of ω, for any $1 \leq k \leq s$. The set of extended admissible cuts of ω is denoted $EAdC(\omega)$.

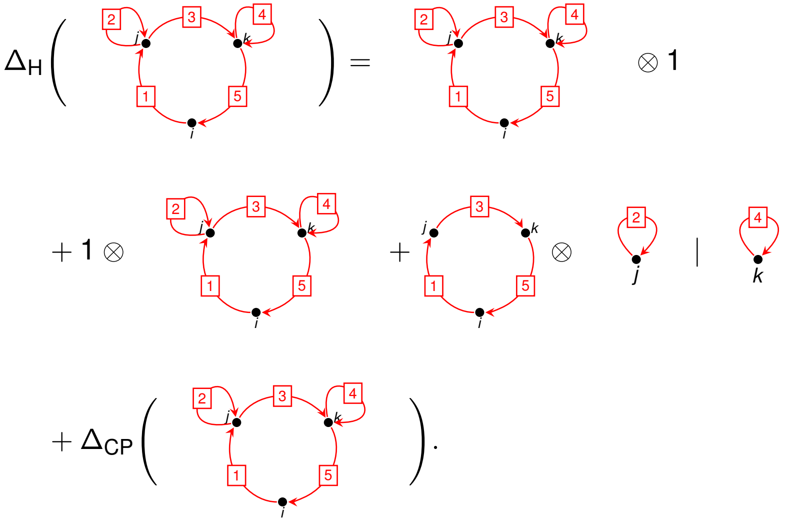
Definition

We define the morphism of algebras Δ_{H} defined by:

$$\Delta_{\mathrm{H}} : \begin{cases} \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle & \longrightarrow & \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle \otimes \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle \\ \omega & \mapsto & \Delta_{\mathrm{H}}(\omega) = \omega \otimes 1 + 1 \otimes \omega \\ & & + \displaystyle\sum_{c \in EAdC(\omega)} \omega_{l_1 l'_1, \dots, l_s l'_s} \otimes \omega^{l_1 l'_1} | \dots | \omega^{l_s l'_s}, \end{cases}$$

where $\omega = w_1 \dots w_m$ is a walk in Γ, the extended admissible cut c is the sequence $1 \leq l_1 < l'_1 < \cdots < l_s < l'_s \leq m$ and the sum is taken over all the extended admissible cuts of ω.

Example



Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ *a finite or countable connected graph. Consider the triple* $(\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle, \star, \Delta_H)$*. It is a Hopf algebra.*

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Let Γ a finite or countable connected graph. Consider the triple $(\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle, \star, \Delta_H)$. It is a Hopf algebra.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

In the graph Γ, we denote by $\mathcal{I}$ the vector space spanned by the elements $\omega_1|\dots|\omega_s - \omega_{\sigma(1)}|\dots|\omega_{\sigma(s)}$ where $\omega_1|\dots|\omega_s \in \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$ and σ is a permutation. Then, $\mathcal{I}$ is a Hopf bi-ideal of $\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$. Thus, $(\mathcal{S}\langle\mathcal{W}(\Gamma)\rangle, \square, \Delta_H)$ is a quotient Hopf algebra of $\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$.

Definition

Let Γ be a finite or a countable connected graph and $\omega = w_1 \dots w_m$ be a walk in Γ such that $AdC(\omega) \neq \emptyset$. For any $(\omega^{kk'}, \omega^{ll'})$ in $AdC(\omega)^2$, the two following statements are equivalent:

1. $\omega^{kk'} \leqslant \omega^{ll'}$.
2. $l \leq k < k' \leq l'$ or $k < k' < l < l'$.

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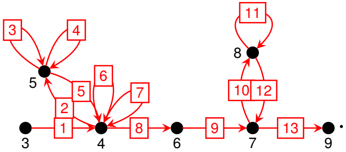
Example

Let consider the walk

$$\psi = 3455544467879 =$$ 

Let $\psi^{35} = 555$, $\psi^{45} = 55$ and $\psi^{11\ 12} = 88$ be three elements in $AdC(\psi)$. Then, $\psi^{45} \leqslant \psi^{35} \leqslant \psi^{11\ 12}$.

Proposition (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let ω be a non-empty finite walk such that $AdC(\omega) \neq \emptyset$. Equipped with the relation $\leqslant$, the set $AdC(\omega)$ is a totally ordered set.

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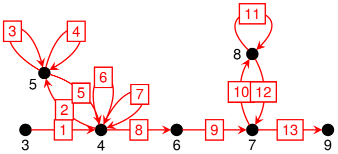
Let ω be a non-empty finite walk such that $AdC(\omega) \neq \emptyset$. Equipped with the relation $\leqslant$, the set $AdC(\omega)$ is a totally ordered set.

Example

Consider the walk

$\psi = 34555444678879 =$



.

Then

$$AdC(\psi) = \{\psi^{45} \leqslant \psi^{35} \leqslant \psi^{78} \leqslant \psi^{68} \leqslant \psi^{28} \leqslant \psi^{11\ 12} \leqslant \psi^{10\ 13}\}.$$

Definition

Let ω be a non-empty finite walk, $AdC(\omega)$ be the set of its admissible cuts and $x(\omega) \in \mathbb{N}$ be the cardinality of $AdC(\omega)$. We assume $AdC(\omega) \neq \emptyset$. Let $s \in \{1, ..., x\}$ be a positive integer and $(\omega^{l_1 l'_1}, \ldots, \omega^{l_s l'_s})$ be a s-tuple of distinct admissible cuts of ω such that $\omega^{l_1 l'_1} \leqslant \cdots \leqslant \omega^{l_s l'_s}$. We associate to this s-tuple a tensor $T_{l_1 l'_1, \ldots, l_s l'_s}$ as follows:

$$T_{l_1 l'_1, \ldots, l_s l'_s} = \omega_{l_1 l'_1, \ldots, l_s l'_s} | \, \omega^{l_s l'_s}_{l_1 l'_1, \ldots, l_{s-1} l'_{s-1}} | \cdots | \omega^{l_i l'_i}_{l_1 l'_1, \ldots, l_{i-1} l'_{i-1}} | \cdots | \, \omega^{l_1 l'_1}.$$

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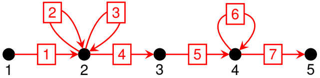
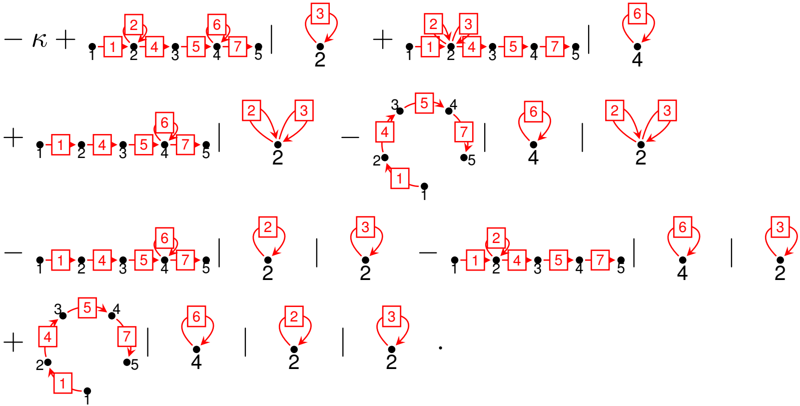
Theorem

Let Γ be a finite connected graph and ω a non-empty finite walk in Γ. Then, in $\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$, the antipode $S(\omega)$ calculated on ω is given by:

$$S(\omega) = -\omega - \sum_{s=1}^{x(\omega)} \sum_{\substack{\omega^{l_1 l'_1} \leqslant \cdots \leqslant \omega^{l_s l'_s} \\ \forall i,\, \omega^{l_i l'_i} \in AdC(\omega) \\ \forall i \neq j,\, \omega^{l_i l'_i} \neq \omega^{l_j l'_j}}} (-1)^s \, T_{l_1 l'_1, \ldots, l_s l'_s}$$

Example

Consider the walk $\kappa = 12223445 =$  . Then $AdC(\kappa) = \{\kappa^{34} \leqslant \kappa^{24} \leqslant \kappa^{67}\}$ and the antipode of κ is

$S(\kappa) = -\kappa +$ 

Pre-Lie product

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The pre-Lie product on walks is given by the linear part in the dual product of Δ_H.

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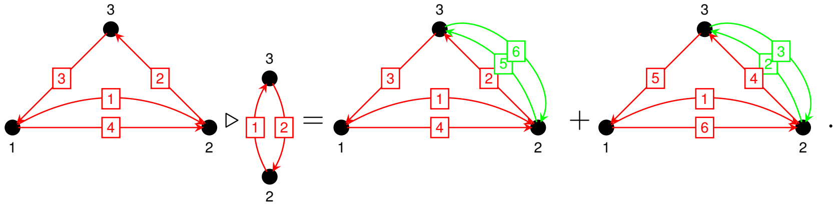

$$12312 \triangleright 232 = 1231232 + 1232312.$$

There are two terms in the result.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let ω walk in a connected finite or countable graph Γ. There exists a unique integer p, a unique p-tuple $(\omega_1, \ldots, \omega_p)$ of walks such that:

$$\omega = \mathit{Parentheses}((\omega_1, \omega_2, \ldots, \omega_p), \triangleright).$$

Besides:

1. *The reconstruction is essentially unique,*
2. $\omega_2, \ldots, \omega_p$ *are simple cycles,*
3. ω_1 *is a self-avoiding walk or a simple cycle.*

Reconstruction of the identity

Questions:

1. How find the integer p and the walks $\omega_1, \dots, \omega_p$?
2. How find the couples of parentheses?

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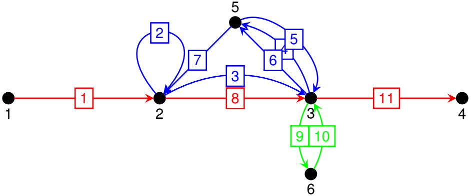
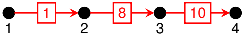
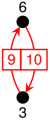
2. How find the couples of parentheses?

Answer: You have to use the total order $\leqslant$ on the set $AdC(\omega)$.

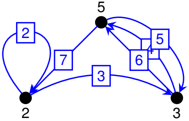
Consider the walk $\omega =$ 

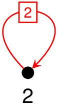
.

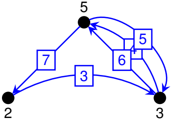
Then: $\omega_1 =$ 

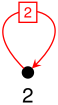
$\omega_2 =$ 

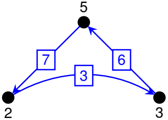
. And then,

$\tau =$ 

$=$ 

$\triangleright$ 

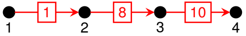
$=$ 

$\triangleright \Bigg($ 

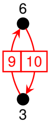
$\triangleright$ 

$\Bigg)$.

Finally:

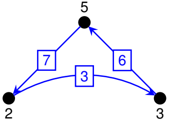
$\omega_1 =$ 

,

$\omega_2 =$ 

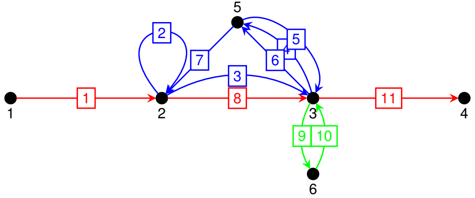
$\omega_3 =$ 

,

$\omega_4 =$ 

$\omega_5 =$ 

.



$= (\omega_1 \triangleright \omega_2) \triangleright (\omega_3 \triangleright (\omega_4 \triangleright \omega_5)).$