Algebraic structures on graph associahedra

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Contents

Algebra and coalgebra structures on graph associahedra

Joint with S. Forcey, *Algebraic structures on graph associahedra* arXiv:1910.00670.

L. Berry, S. Forcey, M. Ronco and P. Showers, *Species substitution, graph suspension, and graded Hopf algebras of painted tree polytopes*, Algebraic & Geometric Topology 19, Issue 2 (2019) 1019-1078

Given a simple finite graph Γ, with n nodes, M. Carr and S. Devadoss defined a poset $Tub(\Gamma)$, whose geometric realization is a convex polytope $\mathcal{K}\Gamma$ of dimension n-1. There exist generalizations of this construction to hypergraphs and finite CW- complexes.

The vector spaces spanned by the faces of certain families of polytopes, give another description of the Hopf algebras of faces of permutahedra and associahedra, as well as the of the free diassociative algebra on one element.

Find out some ideas to describe algebraic structures in this context.

Graph associahedra

M. Carr, S. Devadoss, *Coxeter complexes and graph associahedra*, Topol. and its Applic. 153 (1-2) (2006) 2155-2168.

Definition Γ a simple graph with $\mathrm{Nod}(\Gamma) = \{1, \dots, n\} =: [n]$.

A *tube* t is a subset of $\mathrm{Nod}(\Gamma)$ whose induced graph is a connected subgraph Γ_t of Γ.

Two tubes u and v may interact on the graph as follows:

1. Tubes are *nested* if $u \subset v$.
2. Tubes are *far apart* if $u \cup v$ is not a tube in Γ, that is, the induced subgraph of the union is not connected, (equivalently none of the nodes of u are adjacent to a node of v).

Tubes are *compatible* if they are either nested or far apart.

Γ itself the *universal tube*, we call it t_Γ.

Definition A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible.
When Γ is connected, we assume that every tubing of G to contain (by default) its universal tube.
By the term k-*tubing* we refer to a tubing made up of k tubes, for $k \varepsilon \{1, \ldots, n\}$.
A tubing T of Γ has at most n tubes if Γ has n nodes. A tubing with k tubes is called a k-*tubing* of Γ, for $0 \leq k \leq$ n-1.

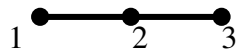
$Tub(\Gamma)$ is partially ordered by the relation:
$T \prec T'$ if T is obtained from T' by adding tubes.

M. Carr and S. Devadoss: the geometric realization of the poset $(Tub(\Gamma), \prec)$ is the barycentric division of a simple, convex polytope $\mathcal{K}\Gamma$ of dimension n-1, whose faces of dimension r are indexed by the n-r tubings of Γ, for $0 \leq r \leq n-1$.

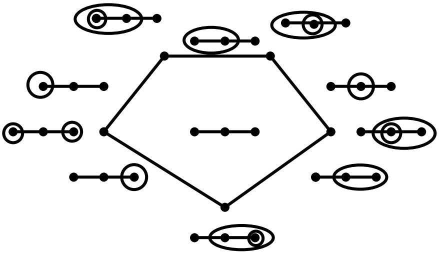
Examples

1. For the graph S_n with $\mathrm{Edg}(S_n) = \phi$, $\mathcal{K}S_n$ is the standard (n-1)-simplex
2. For the linear graph L_n with $\mathrm{Edg}(L_n) = \{(i, i+1) \mid 1 \leq i \leq n-1\}$, $\mathcal{K}L_n$ is the Stasheff polytope of dimension $n-1$, whose faces are indexed by all planar rooted trees with $n+1$ leaves.
3. For the cyclic graph C_n with $\mathrm{Edg}(C_n) = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(1, n)\}$, $\mathcal{K}C_n$ is the cyclohedron of dimension $n-1$.
4. For the complete graph K_n with $\mathrm{Edg}(K_n) = \{(i, j) \mid 1 \leq i < j \leq n\}$,$\mathcal{K}K_n$ is the permutohedron of dimension $n-1$.

For $L_3 =$

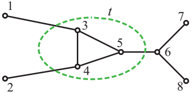


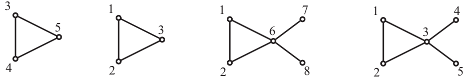
$\mathcal{K}L_3$ is given by



Reconnected graphs

Given a graph Γ and a tube t, construct a new graph Γ^*_t, called the *reconnected complement* : $\mathrm{Nod}(\Gamma^*_t) = \mathrm{Nod}(\Gamma) \setminus \{t\}$ is the set of nodes of Γ^*_t,. There is an edge between two vertices a and b of Γ^*_t if either $\{a, b\}$ to $\{a, b\} \cup t$ is connected.





Basic result The faces of dimension $n-2$ of $\mathcal{K}\Gamma$ are given by $\mathcal{K}\Gamma|_t \times \mathcal{K}\Gamma^*_t$, for a tube t in Γ.

Lemma Let Γ be a graph. For any pair of tubes t and t' in Γ, we have that $(\Gamma_t^*)_{t'}^* = (\Gamma_{t'}^*)_t^*$.

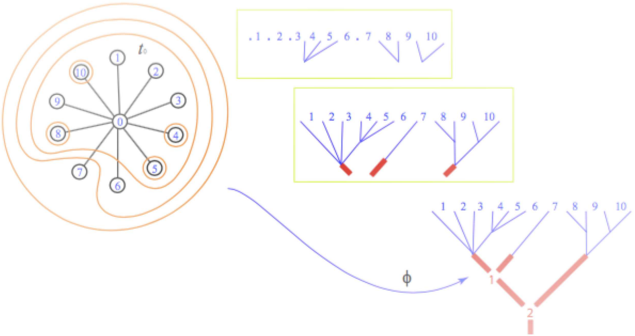
The Lemma justifies the notation $\Gamma^*_{t^1,\dots,t^k}$, for a family of tubes $t^1, \dots, t^k$ in Γ.

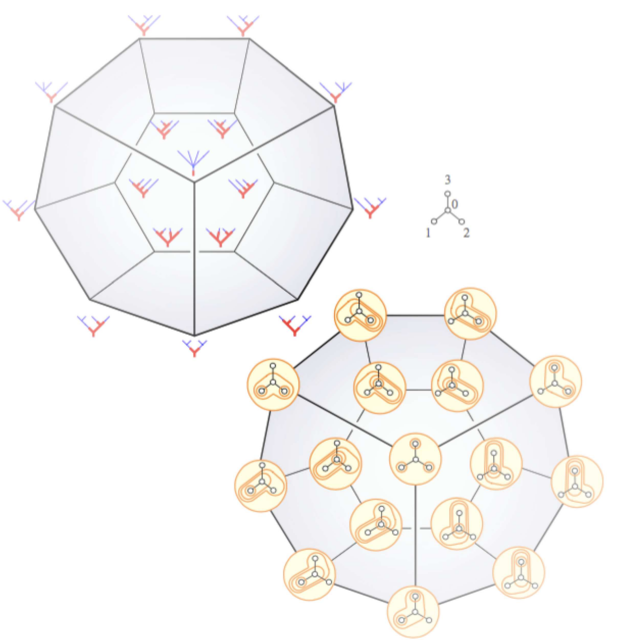
Remark When Γ is the complete graph, respectively the line graph L_n, any tube t in K_n is also complete graph K_r , respectively a line graph L_r, for some $r < n$. Moreover, the reconnected complement $(K_n)^*_{K_r} = K_{n-r}$, respectively $(L_n)^*_{L_r} = L_{n-r}$. For $\Gamma = C_n$, any tube is $t = \{i\}$ and $(C_n)^*_t = C_{n-1}$.

Suspension and colored trees

The *suspension* of Γ is the graph obained by adding a node 0 and create on edge $\{0, i\}$ for any node $i \,\varepsilon\, \Gamma$.
More generally $\Gamma \vee \Omega$ is the graph which has Γ and Ω as subgraphs, and where any node of Γ is linked to any node of Ω.

Suspension induces the notion of colored trees




3
0
1
2

PreLie coproduct

The co-preLie coproduct is defined by:

$$\Delta(T) := \sum_{t \,\varepsilon\, T} T|_{\Gamma_t} \otimes T|_{\Gamma_t^*}.$$

for $T \,\varepsilon\, \mathrm{Tub}(\Gamma)$, where we consider the empty tube $t_{\emptyset}$ and the universal tube t_Γ in order to get units.
Δ satisfies that

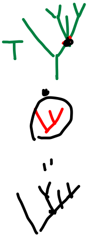
$$\underline{(\Delta \otimes \mathrm{Id}) \circ \Delta - (\mathrm{Id} \otimes \Delta) \circ \Delta} = (\tau \otimes \mathrm{Id}) \circ ((\Delta \otimes \mathrm{Id}) \circ \Delta - (\mathrm{Id} \otimes \Delta) \circ \Delta).$$

Δ is coassociative on the vector subspace spanned by the tubings of complete graphs, it gives the coproduct on the faces of permutohedra.

Substitution

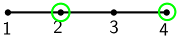
Definition Let Γ be a graph and let T and T' be tubings of Γ, and let $t \varepsilon T$ be a tube.

1. Let S be a tubing of Γ_t which is compatible with $T|_t$. The tubing $T \circ_t S$ is the set of tubes s satisfying that $s \varepsilon T$, or $s \varepsilon S$ (considered as a tube in Γ).
2. Given a tubing S of Γ^*_T, define the *substitution* of S in T to be the tubing $T \bullet_\Gamma S$ of Γ whose elements are the tubes t satisfying one of the following conditions:
   - 2.1 t belongs to T,
   - 2.2 $t = s \coprod t^{i_1} \cdots \coprod t^{i_k}$, where s is a tube of S and $\{t^{i_1}, \ldots, t^{i_k}\}$ is the set of maximal proper tubes of T which are linked to at least one of a union of tubes which comprise s in Γ.

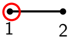


Definition

Let T be a tubing of Γ. For any tube $t \varepsilon T$ and any tubing $S \varepsilon \mathrm{Tub}((\Gamma_t)^*_{T|_t})$, the *t-substitution* of S in T is the tubing $T \circ_t (T|_t \bullet_{\Gamma_t} S)$ on Γ. We denote it simply by $\gamma_t(T; S)$.

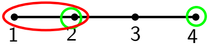


$T = \{\{2\}, \{3\}\}$

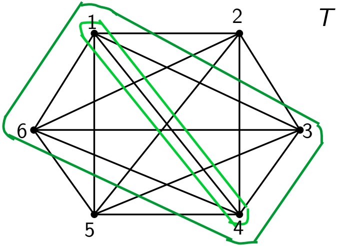


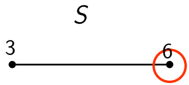
$\Gamma^*_T = L_2$ and $S = \{\{1\}\}$

we get

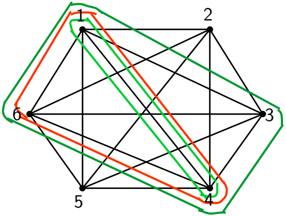


$\gamma_{t_{L_4}}(T; S)$





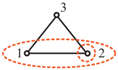
where $S = \{\{2\}\}$ is a tubing in $(\Gamma_{\{1,3,4,6\}})^*_{T|_{\{1,3,4,6\}}} = K_2$. We get that $\gamma_{\{\{1,3,4,6\}\}}(T; S) =$



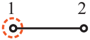
Substitution on complete graphs

As tubings in K_8

$$S^0 = \{\{2\}, \{1,2\}\}$$



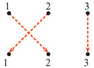
$$S^1 = T_{\emptyset} \text{ and } S^2 = \{\{2\}, \{1,2\}, t_{K_3}\}$$



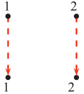
$$S^3 = \{\{1\}, \{1,2\}\}$$

As surjective maps

$$x_{S^0} = (2,1)$$

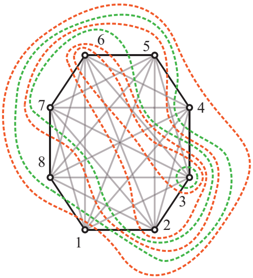


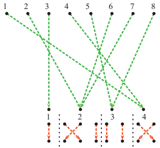
$$x_{S^1} = (1) \text{ and } x_{S^2} = (2,1,3$$



$$x_{S^3} = (1,2)$$

the substitution $\gamma(T; S^0, S^1, S^2, S^3)$ is given by:





$$x_{\gamma(T;S^0,S^1,S^2,S^3)} = (8,3,1,7,5,2,4,6),$$

M. Ronco, *Shuffle bialgebras*, Ann. Inst. Fourier, Vol. 61, 3 (2011) 799-850.
J.-L. Loday, M. Ronco, *Permutads*, J. of Comb. Theory, Series A 120 (2013) 340–365.

Theorem (Associativity of substitution) Let T be a tubing in Γ and let t be a tube in T. Given a tubing $S \varepsilon \mathrm{Tub}((\Gamma_t)^*_{T|_t})$ and a proper tube $s \varepsilon S$, we have that:

1. The tube s induces a tube $\tilde{s} \subsetneq t$ in Γ, given by $\tilde{s} = s \cup \{t_{i_1}, \ldots, t_{i_r}\}$, where $t_{i_1}, \ldots, t_{i_r}$ are the maximal proper tubes of $T|_t$ which are linked to some node of s.
2. The graphs $(((\Gamma_t)^*_{T|_t})_s)^*_{S|_s}$ and $(\Gamma_{\tilde{s}})^*_{\gamma_t(T,S)|_{\tilde{s}}}$ are equal.
3. For any tubing W of $(((\Gamma_t)^*_{T|_t})_s)^*_{S|_s}$, the tubing $\gamma_s(S; W)$ is a tubing of $(\Gamma_t)^*_{T|_t}$, which satisfies that

$$\gamma_t(T, \gamma_s(S, W)) = \gamma_{\tilde{s}}(\gamma_t(T, S), W).$$

Note that Theorem implies that the substitution γ is associative in the following way:
Let $T = \{t_\Gamma = t^0, t^1, \dots, t^k\}$ be a tubing in Γ, and let $S^i = \{s^{0i}, \dots, s^{l_i i}\}$ be a family of tubings $S^i \, \varepsilon \, \mathrm{Tub}((\Gamma_{t^i})^*_{T|_{t^i}}$, for $0 \leq i \leq k$. Suppose that $W^{ji} \, \varepsilon \, \mathrm{Tub}(\Gamma_{s^{ji}})^*_{T|_{s^{ji}}}$ is a collection of tubings, for each pair (i, j) with $0 \leq j \leq i$. We have that:

$$\gamma(\gamma(T;S^0, \dots, S^k); W^{00}, \dots, W^{l_0 0}, \dots, W^{l_k k}) =$$
$$\gamma(T; \gamma(\gamma_{t^0}(T, S^0)|_{t^0}; W^{00}, \dots, W^{l_0 0}), \dots,).$$

Proposition Let Ω be a subgraph of Γ with the same set of nodes. For any $T \varepsilon \text{Tub}(\Gamma)$, any tube $t \varepsilon T$ and any tubing $S \varepsilon \text{Tub}(\Gamma^*_{T|_t})$, we have that:

$$\gamma(\text{res}^\Gamma_\Omega(T); \tilde{S}^0, \dots, \tilde{S}^k) = \text{res}^\Gamma_\Omega(\gamma_t(T, S)),$$

where the tube t induces a tubing $\text{res}^\Gamma_\Omega(t) = \{t_1, \dots, t_k\}$, with $t_i \cap t_j = \phi$ for $i \neq j$, and we denote by $\tilde{S}^i$ the tubing induced by S on the reconnected complement $(\Omega_{t_i})^*_{\text{res}^\Gamma_\Omega(T)|_{t_i}}$.

Proposition For any connected graph Γ, a tubing $T \varepsilon \text{Tub}(\Gamma)$ may be obtained from $(\Gamma, \{t_\Gamma\})$ applying substitutions of type $\gamma_{t_\Gamma}(\ , \{t\})$, where $\{t\}$ denotes the tubing whose unique tubes are t_Γ and t.

Graph associahedra described by operations and relations

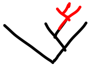
For a tube $t = \{i_1 < \cdots < i_{|t|}\}$ in Γ, where $|t|$ denotes the number of nodes in t, we denote $\sigma_t \,\varepsilon\, \Sigma_n$ the permutation whose image is

$$\sigma_t := (i_1, \ldots, i_{|t|}, j_1, \ldots, j_{n-|t|}),$$

where $\mathrm{Nod}(\Gamma_t^*) = [n] \setminus \mathrm{Nod}(t) = \{j_1 < \cdots < j_{n-|t|}\}$.

The binary operation $\circ_{(\Gamma,t)}$ is partially defined on **Tub**, as follows:

$$S \circ_{(\Gamma,t)} W := \gamma_{t_\Gamma}(T_\Gamma \circ_t S, W),$$



for any pair of tubings $S \,\varepsilon\, \mathrm{Tub}(\Gamma_t)$ and $W \,\varepsilon\, \mathrm{Tub}(\Gamma_t^*)$. That is, a tube $w \,\varepsilon\, T_\Gamma \circ_t S$ is either t_Γ or a tube in S.

Proposition The operations $\circ_{(\Gamma,t)}$ satisfy the following relations:

1. For two tubes t and t' in a graph Γ, which are not linked, we get that:
$$T_2 \circ_{(\Gamma,t')} (T_1 \circ_{(\Gamma^*_{t'},t)} S) = T_1 \circ_{(\Gamma,t)} (T_2 \circ_{(\Gamma^*_t,t')} S),$$
for $T_1 \varepsilon \mathrm{Tub}(\Gamma_t)$, $T_2 \varepsilon \mathrm{Tub}(\Gamma_{t'})$ and $S \varepsilon \mathrm{Tub}(\Gamma^*_{t,t'})$,
2. For two tubes $t' \subsetneq t$ in a graph Γ,
$$(T_2 \circ_{(\Gamma_t,t')} T_1) \circ_{(\Gamma,t)} S = T_2 \circ_{(\Gamma,t')} (T_1 \circ_{(\Gamma^*_{t'},\tilde{t})} S),$$
for $T_1 \varepsilon \mathrm{Tub}((\Gamma_t)^*_{t'})$, $T_2 \varepsilon \mathrm{Tub}(\Gamma_{t'})$ and $S \varepsilon \mathrm{Tub}(\Gamma^*_t)$, where $\tilde{t}$ denotes the tube induced by t in $\Gamma^*_{t'}$.

Theorem Let CGraph be the set of all graded simple connected finite graphs. The vector space **Tub**, equipped with the partially defined binary operations $\circ_{(\Gamma,t)}$, for any $\Gamma \,\varepsilon$ CGraph and any tube t in Γ, is the free object spanned by the set CGraph and the products $\circ_{(\Gamma,t)}$.

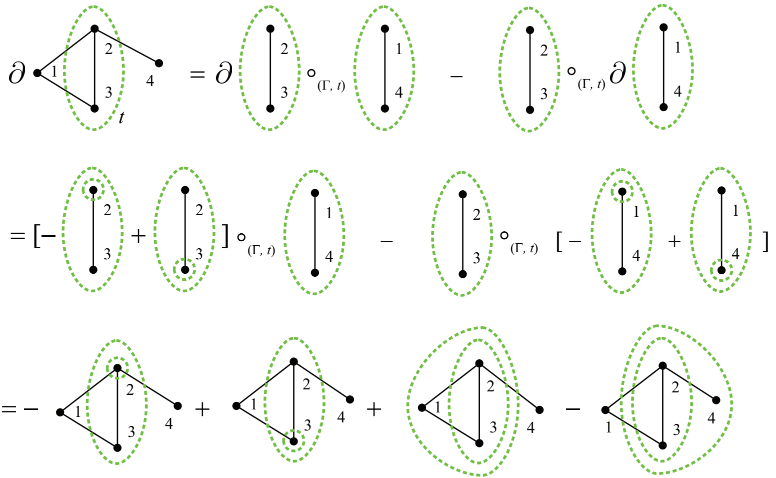
Definition Let Γ be a finite connected simple graph, define the map $\partial : \mathbb{K}[\mathrm{Tub}(\Gamma)] \longrightarrow \mathbb{K}[\mathrm{Tub}(\Gamma)]$, , as the unique $\mathbb{K}$-linear endomorphism satisfying:

1. $\partial(T_\Gamma) = \sum_{t \subsetneq \Gamma} (-1)^{|t|} \mathrm{sgn}(\sigma_t)\{t\}$,
   where the sum is taken over all the tubes t is Γ different from the universal tube t_Γ.
2. For any tube t in Γ and any pair of tubings $T \,\varepsilon\, \mathrm{Tub}(\Gamma_t)$ and $S \,\varepsilon\, \mathrm{Tub}(\Gamma_t^*)$,

$$\partial(T \circ_{(\Gamma,t)} S) = \partial(T) \circ_{(\Gamma,t)} S + (-1)^{|t|} T \circ_{(\Gamma,t)} \partial(S).$$

The Theorem shows that there exists a unique linear map ∂ satisfying both conditions.

Example



Weak facial order

Consider the extension of the weak Bruhat order to all faces of permutahedra, introduced in

1. D. Krob, M. Latapy, J.-C. Novelli, Ha-D. Phan, S. Schwer, *PseudoPermutations I: First Combinatorial and Lattice Properties*, 13th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2001), 2001.
2. P. Palacios, M. Ronco, *Weak Bruhat order on the set of faces of the permutohedron and the associahedron*, J. Algebra, 299(2) 648-678, 2006.

Studied and generalized by A. Dermenjian, C. Hohlweg, V. Pilaud, *The facial weak order and its lattice quotients*, 2016.

Let Γ be a connected simple finite graph with n nodes. The weak facial order induces a partial order on the faces of graph associahedra, given by the following conditions

1. Let $T_{\leq i}$ be the minimal tubing containing the nodes $1, \dots, i$, for some $1 \leq i < n$, then $T_{\leq i} <_f \{t_\Gamma\}$,
2. Let $T_{\geq i}$ be the minimal tubing containing the nodes $i, \dots, n$, for some $1 < i \leq n$, then $\{t_\Gamma\} <_f T_{\geq i}$,
3. If $T <_f T'$ are tubings of Γ, such that $\Gamma^*_T = \Gamma^*_{T'}$, and S is a tubing in Γ^*_T, then $T \bullet_\Gamma S <_f T' \bullet_\Gamma S$,
4. If $S <_f S'$ tubings of Γ^*_T, then $T \bullet_\Gamma S <_f T \bullet_\Gamma S'$.

In this case we get a triangulation of graph associahedra, which extends the one introduced by J.-L. Loday in
Parking functions and triangulation of the associahedron, Proceedings of the Street's fest 2006, Contemp. Math. AMS 431 (2007), 327-340.

Substitution on trees

Proposition For $n \geq 1$ and any tube $t = \{i+1, \dots, i+r\}$ in L_n, there exists a bijective map $\rho_n : \mathrm{Tub}(L_n) \longrightarrow \mathcal{T}r_{n+1}$ satisfies that

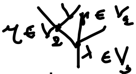
1. for any pair of tubings $S \,\varepsilon\, \mathrm{Tub}(L_r)$ and $W \,\varepsilon\, \mathrm{Tub}((L_n)^*_t) = \mathrm{Tub}(L_{n-r})$,

$$\rho_n(S \circ_{L_n,t} W) = \rho_{n-r}(W) \circ_{i+1} \rho_r(S).$$

2.
$$\rho_n(\gamma_t(T;S)) = \rho_n(T) \circ_{a_t} \rho_r(S),$$

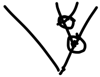
   for any tubing T of L_n and any $S \,\varepsilon\, \mathrm{Tub}((L_r)^*_{T|_t})$, where a_t is the internal vertex of $\rho_n(T)$ associated to the tube t.

$\mathbb{P}$ $V = \bigoplus V_n$



Define a functor $\mathbb{G} : \mathrm{gVect}_{\mathbb{K}} \longrightarrow \mathrm{gVect}_{\mathbb{K}}$ as follows:

1. for any tube t is a tubing T in Γ, define the *arity* of t as the number of nodes in t which do not belong to any other tube $t' \varepsilon T|_t$ plus one. We denote the arity of t by $\mathrm{ar}(t)$.
2. for any tubing T of Γ and any graded vector space V, let $V_T := \bigotimes_{t \,\varepsilon\, T} V_{\mathrm{ar}(t)}$.
3. define



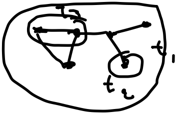
$\mathbb{P}$

$$\mathbb{G}(V)_{n+1} := \bigoplus_{|\mathrm{Nod}(\Gamma)|=n} \left(\bigoplus_{T \,\varepsilon\, \mathrm{Tub}(\Gamma)} V_T \right).$$

$\underline{\mathbb{P} \circ \mathbb{P} \to \mathbb{P}}$

$\underline{Id \to \mathbb{P}}$

$\lambda \in V_n \to$ 



$\mathrm{ar}(t_1) = 3 + 1 = 4$

However, we do not get a monad structure due to:

1. $\mathbb{G}$ is not unital, because there exist many finite simple connected graphs with n vertices, for any fixed $n \geq 3$,
2. the composition $\mathbb{G} \circ \mathbb{G}$ is not always defined. For a graph Γ and a tubing $T \,\varepsilon\, \mathsf{Tub}(\Gamma)$, we get that

$$\mathbb{G}(\mathbb{G}(V))_T = \bigotimes_{t \,\varepsilon\, T} \Big(\bigoplus_{|\mathsf{Nod}(\Omega_t)| = \mathsf{ar}(t)-1} \left(\bigoplus_{S_t \,\varepsilon\, \mathsf{Tub}(\Omega_t)} V_{S_t} \right) \Big).$$

But, in order to apply substitution and get an element in $\mathbb{G}(V)$, we need that $\Omega_t = (\Gamma_t)^*_{\mathsf{Maxt}(T|_t)}$, for any $t \,\varepsilon\, T$.

Batanin-Markl strict operadic category associated to substitution

M. Batanin, M. Markl, *Operadic categories and duoidal Deligne's conjecture*, Adv. in Math 285 (2015), 1630-1687.
R. Kaufmann, B. C. Ward *Feynman categories*, preprint arXiv 1312.1269 (2013)

The category sFSet

Definition The category sFSet is the category whose objects are the linearly ordered sets $[n] = \{1 < 2 < \cdots < n\}$, for $n \geq 1$, and whose arrows are map between finite sets, which do not necessarily preserve the order.

The terminal object of sFSet is $[1]$.

Definition Given two maps $f \,\varepsilon\, \mathrm{sFSet}([n], [m])$ and $i \,\varepsilon\, \mathrm{sFSet}([1], [m])$ the i^{th} *fiber* of f on i is the pull back $f^{-1}(i)$ of the diagram:

$$\begin{array}{ccc} f^{-1}(i) & \longrightarrow & [n] \\ \downarrow & & \downarrow \\ [1] & \longrightarrow & [m] \end{array}$$

where $f^{-1}(i)$ is identified with the finite set $[f^{-1}(i)]$.

Operadic category

Strict operadic categories are categories with a functor to sFSet, having a family of terminal objects (one for each connected component of the category) and certain inverse images, which behave like pull-backs.

A *strict operadic category* is a category $\mathcal{O}$ together with:

- (a) a fixed family of terminal objects U_c, for each connected component $c \, \varepsilon \, \pi_0(\mathcal{O})$,
- (b) a *cardinality* functor $|-| : \mathcal{O} \longrightarrow \mathsf{sFSet}$,
- (c) an object $f^{-1}(i)$ in $\mathcal{O}$ such that $|f^{-1}(i)| = |f|^{-1}(i)$, for every pair of homomorphisms $f \, \varepsilon \, \mathcal{O}(T, S)$ and every element $i \, \varepsilon \, |S|$,

satisfying certain conditions.

Definition Given an strict operadic category $\mathcal{O}$ and a monoidal category $\mathcal{C}$, an *$\mathcal{O}$-collection* in $\mathcal{C}$ is a collection $\{E(T)\}_{T \,\varepsilon\, \mathcal{O}}$ of objects of $\mathcal{C}$, indexed by the objects of the category $\mathcal{O}$. For an $\mathcal{O}$-collection E in $\mathcal{C}$ and a homomorphism $f : T \longrightarrow S$ in $\mathcal{O}$, the object $E(f)$ in $\mathcal{C}$ is defined by

$$E(f) := \bigotimes_{i \,\varepsilon\, |S|} E(f^{-1}(i)).$$

Operadic category associated to graph associahedra

Describe an operadic category $\mathcal{O}_{CD}$ such that the substitution of tubings provides a natural example of $\mathcal{O}_{CD}$ operad. Our model is M. Markl's operadic category Per.

Definition

Define the category $\mathcal{O}_{CD}$ as follows:

1. The objects of $\mathcal{O}_{CD}$ are pairs (Γ, T), where Γ is a connected simple finite graph and T is a tubing of Γ.
2. The homomorphisms in $\mathcal{O}_{CD}$ are given by:

$$\mathcal{O}_{CD}((\Gamma, T), (\Omega, S)) := \begin{cases} \phi\,, & \text{for } \Gamma \neq \Omega \text{ or } T \not\preceq S, \\ \{\iota_{\Gamma,T,S}\}, & \text{for } \Gamma = \Omega \text{ and } T \preceq S, \end{cases}$$

where $T \preceq S$ is Carr and Devadoss order, and means that T is obtained from S by adding compatible tubes.

Note that $\pi_0(\mathcal{O}_{CD}) = \mathrm{CGraph}$, the set of all simple connected finite graphs equipped with a total order on the set of nodes, and the terminal objects of $\mathcal{O}_{CD}$ are (Γ, T_Γ), for $\Gamma \varepsilon \mathrm{CGraph}$.

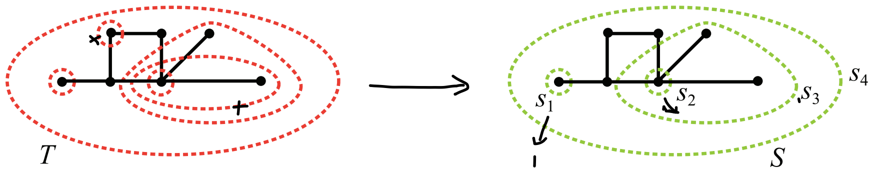
Let Γ be a finite connected simple graph and let T be a tubing of Γ, we denote by $\mathbb{L}(T)$ the number of tubes of T. The definition of a functor from the category of tubings to the category sFSet, requires a standard way to enumerate tubes in a tubing T.

For $f = \iota_{\Gamma,T,S}$ and $i \,\varepsilon\, |(\Gamma, S)|$, there exists a unique tube $s \,\varepsilon\, S$ such that $\mathfrak{N}_S(s) = i$. Define $f^{-1}(i)$ as:

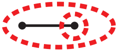
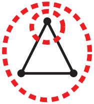
$$f^{-1}(i) = \left((\Gamma_s)^*_{(S|_s)}, (T|_s)^*_{(S|_s)} \right),$$

where $(\Gamma_s)^*_{(S|_s)}$ is the reconnected complement of Γ_s by the set $\mathrm{Maxt}(S|_s)$ of proper maximal tubes of $S|_s$, and $(T|_s)^*_{(S|_s)}$ denotes the tubing induced by $T|_s$ on $(\Gamma_s)^*_{(S|_s)}$.

For example, consider



we get that

$f^{-1}(1) =$  $= f^{-1}(2)$ $\quad f^{-1}(3) =$  $\quad f^{-1}(4) =$ 

Proposition The category $\mathcal{O}_{CD}$ is a strict operadic category .

Thanks!!!