A coupling of rooted spectral measures

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Spectral measure

$A = (a_{ij})_{i,j=1,\ldots,N}$ real symmetric matrix.

Definition

The local spectral measure at i is the unique measure μ_i on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with moments:

$$\int_{\mathbb{R}} x^k d\mu_i(x) = (A^k)_{ii} \ , \ k = 0, 1, 2, \ldots$$

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For A the adjacency matrix of a graph G, μ_i's k-th moment counts the length k closed walks from i to i in G.

In this wase, we call μ_i the *rooted spectral measure* at i.

Spectral measure

Proposition

Let $A = P\Lambda P^\top$ be a spectral decomposition of A with $P = (p_{ij})_{i,j=1,\ldots,N}$ orthogonal and $\Lambda = \mathrm{Diag}(\lambda_1, ..., \lambda_N)$, alors

$$\mu_i = \sum_{j=1}^{N} p_{ij}^2 \ \delta_{\lambda_j}.$$

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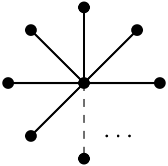
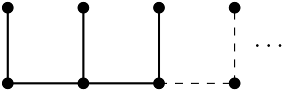
- $(A^k)_{ii} = (P\Lambda^k P^\top)_{ii} = \sum_{j=1}^{N} p_{ij}^2 \lambda_j^k$
- Does not depend on the choice of the spectral decomposition
- Spectral measure $\sum_{i=1}^{N} \delta_{\lambda_i} = \sum_{i=1}^{N} \mu_i$ (k-th moment $= \mathrm{tr}(A^k)$)

In the literature

- Asymptotic distribution of random matrices spectra
- Benjamini-Schramm convergence of rooted random graphs
- Limit theorems on star graphs and comb graphs





- Reconstruction conjecture: Is a (non-regular) graph characterized by its rooted spectral measures?

Coupling

Goal: Define an "interesting" joint measure μ on $\mathbb{R}^N$ with margins $\mu_1, ..., \mu_N$

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Goal: Define an "interesting" joint measure μ on $\mathbb{R}^N$ with margins $\mu_1, ..., \mu_N$

Let S_N be the set of permutations on $\{1, ..., N\}$ and π the signed measure on S_N defined by

$$\pi(\{\sigma\}) = \epsilon(\sigma) \prod_{j=1}^{N} p_{j\sigma(j)} \ , \ \sigma \in S_N$$

where $\epsilon(.)$ is the signature.

- $\pi(\sigma) = \det(P \odot M_\sigma)$ with M_σ the permutation matrix associated to σ and $\odot$ the Hadamard product.
- $\pi(S_N) = \sum_{\sigma \in S_N} \pi(\{\sigma\}) = \det(P) = 1$ (quasi-probability)

For $\lambda = (\lambda_1, ..., \lambda_N)$ the spectrum of A, let

$$\lambda_\sigma = (\lambda_{\sigma(1)}, ..., \lambda_{\sigma(n)}) \ , \ \sigma \in S_N .$$

Definition

The *joint spectral measure* μ is the pushforwd measure of π by $\sigma \mapsto \lambda_\sigma$:

$$\mu(\{\lambda_\sigma\}) = \sum_{\tau:\lambda_\tau=\lambda_\sigma} \epsilon(\tau) \prod_{j=1}^{N} p_{j\tau(j)}$$

- $\mathrm{Supp}(\mu) = \{\lambda_\sigma \ , \ \sigma \in S_N\}$
- Distinct eigenvalues: $\mu(\{\lambda_\sigma\}) = \pi(\{\sigma\}) = \epsilon(\sigma) \prod_{j=1}^{N} p_{j\sigma(j)}$
- Does not depend on the spectral decomposition
- μ is a *quasi-probability*: $\mu(\{\lambda_\sigma \ , \ \sigma \in S_N\}) = 1$

Multivariate moments

let

- $m[k_1, ..., k_N]$ is the multivariate moment

$$m[k_1, ..., k_N] = \int_{\mathbb{R}^N} x_1^{k_1} ... \, x_N^{k_N} \, d\mu(x_1, ..., x_N).$$

- $A[k_1, ..., k_N]$ is the $N \times N$ whose i-th column is the i-th column oof A^{k_i}, $i = 1, ..., N$:

$$A[k_1, ..., k_N] = \left[\begin{bmatrix} (A^{k_1})_{11} \\ \vdots \\ (A^{k_1})_{N1} \end{bmatrix} \cdots \begin{bmatrix} (A^{k_N})_{1N} \\ \vdots \\ (A^{k_N})_{NN} \end{bmatrix} \right]$$

Lemma

For all $k_1, ..., k_N \in \mathbb{N}$,

$$m[k_1, ..., k_N] = \det\left(A[k_1, ..., k_N]\right).$$

Proof:

- $m[k_1, ..., k_N] = \sum_{\sigma \in S_N} \lambda_{\sigma(1)}^{k_1} ... \lambda_{\sigma(n)}^{k_N} \mu(\{\lambda_\sigma\})$

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- $A^k P = P\Lambda^k \implies (A^k P)_{ij} = p_{ij}\lambda_j^k$ and

$$m[k_1, ..., k_N] = \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{i=1}^{N} (A^{k_i} P)_{i\sigma(i)} = \det(A[k_1, ..., k_N]P) \quad \square$$

Corollary

The margins of μ are the rooted spectral measures μ_i.

Proof:
μ_i is characterized by its moments, with for all $k \in \mathbb{N}$:

$$\int_{\mathbb{R}} x^k d\mu_i(x) = (A^k)_{ii} = \det\left(A[0, ..., 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{k}, 0, ..., 0]\right) \qquad \square$$

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Let $X = (X_1, ..., X_N)$ be a vector with (quasi)-distribution μ, we write

$$\mathbb{E}(g(X)) = \int_{\mathbb{R}^N} g(x) d\mu(x) \quad \text{e.g.} \quad m[k_1, ..., k_N] = \mathbb{E}(X_1^{k_1} ... X_N^{k_N})$$

Proposition

If A is the adjacency matrix of a simple graph,

$$\mathrm{var}(X) = L$$

where L is the graph Laplacian.

Preuve:

- $\mathrm{var}(X_i) = (A^2)_{ii} = \deg(i)$
- $\mathrm{cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)$

$$= \det\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ij} & 0 \end{bmatrix}\right) = -a_{ij}^2 = -a_{ij} \quad (i \neq j) \qquad \square$$

More generally: $\mathrm{cov}(X_i^k, X_j^k) = -(A^k)_{ij}^2$

Probabilities on subsets

For $u, v \subset \{1, ..., N\}$, let M_{uv} be the submatrix of M restricted to $(i, j), i \in u, j \in v$.

Proposition

If the eigenvalues λ_j are distinct, for all subsets $u, v \subset \{1, ..., N\}$ of equal sizes,

$$\mathbb{P}\big(\{X_i, i \in u\} = \{\lambda_j, j \in v\}\big) = \det(P_{uv})^2.$$

- If u, v are singletons,
$$\mathbb{P}(X_i = \lambda_j) = p_{ij}^2$$
- The proof uses the Jacobi identity
$$\det(P_{\overline{uu}}) = \det(P^{-1}_{\overline{uu}}) = \frac{\det(P_{uu})}{\det(P)}$$
where $\overline{u} = \{1, ..., N\} \setminus u$.

Weak convergence

Let $\{G_n, n \in \mathbb{N}\}$ be a sequence of nested graphs built by merging n copies of a graph G at a commun vertex o:

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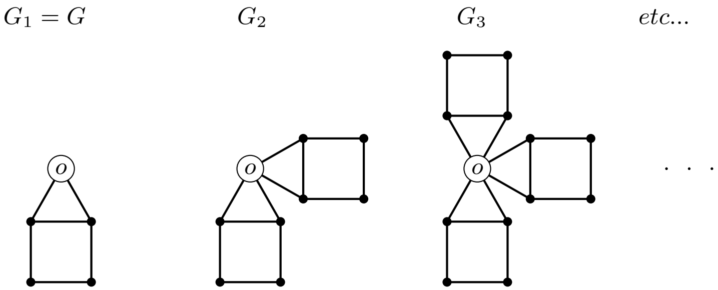


Limit theorem

Theorem (Obata 2004)

Let $\mu_o^{(n)}$ be the spectral measure of G_n rooted at o ,

$$\sqrt{n}\mu_o^{(n)}(\sqrt{n}\ .) \underset{n\to\infty}{\longrightarrow} \frac{1}{2}(\delta_{-\sqrt{d_o}} + \delta_{\sqrt{d_o}})$$

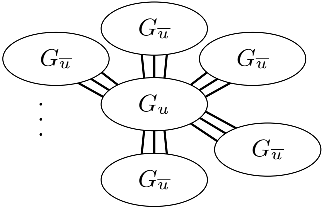
where d_o is the degree of o in G.

- Only depends of the immediate neighborhood in G (through d_o)
- Letting $X_o^{(n)} \sim \mu_o^{(n)}$: $X_o^{(n)}/\sqrt{n} \xrightarrow[n\to\infty]{loi} \sqrt{d_o}B$ with $B \sim \mathrm{Rad}(1/2)$

Generalization

The $G_n, n \in \mathbb{N}$ are built by merging n copies of G rooted at a commun subgraph G_u of p vertices $u = \{1, ..., p\}$:

Graph G_n



Adjacency matrix $A^{(n)}$

$$\begin{bmatrix} A_{uu} & A_{u\overline{u}} & \dots & A_{u\overline{u}} \\ A_{\overline{u}u} & A_{\overline{u}\overline{u}} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ A_{\overline{u}u} & 0 & 0 & A_{\overline{u}\overline{u}} \end{bmatrix}$$

Theorem

Let $\mu^{(n)}$ be the joint spectral measure G_n and $(X_1^{(n)}, ..., X_p^{(n)})$ a vector of the p quasi-random variables rooted at u,

$$\frac{\left(X_1^{(n)}, ..., X_p^{(n)}\right)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \left(B_1\sqrt{Y_1}, \cdots, B_p\sqrt{Y_p}\right),$$

where

- $(Y_1, ..., Y_p)$ is a quasi-random vector with distribution the joint spectral measure of $D = A_{u\overline{u}}A_{\overline{u}u}$
- $B_1, ..., B_p$ are iid Rademacher Rad$(1/2)$, independent from the Y_i's

$D_{ij} = \#\{\text{neighbors in } G_{\overline{u}} \text{ to both } i \text{ and } j\}$, $i, j \in u$

Proof: (convergence of normalized moments)

- $m_u^{(n)}[k_1, ..., k_p] = \mathbb{E}\left[\left(\frac{X_1^{(n)}}{\sqrt{n}}\right)^{k_1} ... \left(\frac{X_p^{(n)}}{\sqrt{n}}\right)^{k_p}\right] = \det\left(\frac{A^{(n)}}{\sqrt{n}}[k_1, ..., k_p, 0, ...]\right)$

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- $$I + zA^{(n)} + z^2 {A^{(n)}}^2 + ... = \begin{bmatrix} I - zA_{uu} & A_{u\overline{u}} & \dots & A_{u\overline{u}} \\ A_{\overline{u}u} & I - zA_{\overline{uu}} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ A_{\overline{u}u} & 0 & 0 & I - zA_{\overline{uu}} \end{bmatrix}^{-1}$$

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- Schur complement:

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}^{-1} = \begin{bmatrix} \left(A - BC^{-1}B^\top\right)^{-1} & . \\ . & . \end{bmatrix}$$

- $\left(\left(I - zA^{(n)}\right)^{-1}\right)_{uu} = \left(I - zA_{uu} - nz^2 A_{u\overline{u}}\left(I - zA_{\overline{uu}}\right)^{-1} A_{\overline{u}u}\right)^{-1}$

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- $\left(\left(I - \frac{z}{\sqrt{n}}A^{(n)}\right)^{-1}\right)_{uu} \underset{n\to\infty}{\longrightarrow} (I - z^2 D)^{-1}$
- $m_u^{(n)}[k_1, ..., k_p] \underset{n\to\infty}{\longrightarrow} \begin{cases} \det\left(D\left[k_1/2, ..., k_p/2\right]\right) & \text{if } k_1, ..., k_p \text{ are even} \\ 0 & \text{sinon} \end{cases}$

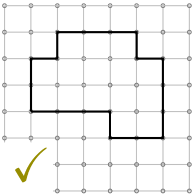
Some combinatorics properties

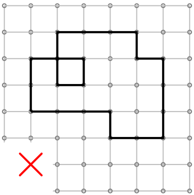
Let A be the adjacency matrix of a graph G

- $(A^k)_{ij}$ is the number of length k walks from i to j in G
- Matrix generating function of walks in G:

$$z \mapsto (I - zA)^{-1} = I + zA + z^2A^2 + ...$$

Let $\mathcal{C}$ be the set of simple cycles on G





Let $V(c)$ the set of vertices met by c and $|c|$ its length,

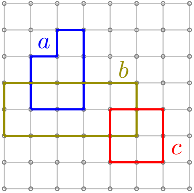
$$c \in \mathcal{C} \iff \#\big(V(c)\big) = |c|$$

Characteristic polynomial

$$\det(I - zA) = \sum_{p\geq 0}(-1)^p \sum_{\substack{c_1,\dots,c_p\in\mathcal{C} \\ V(c_i)\cap V(c_j)=\emptyset, i\neq j}} z^{|c_1|+\dots+|c_p|}$$

We denote by $\mathcal{H}$ the set of "words" formed by simple cycles, under the commutation rule

$$cc' = c'c \iff V(c)\cap V(c') = \emptyset.$$



$abc \neq acb$

$acb = cab$

$bca = bac$

$etc...$

We call *hike* an element of $\mathcal{H}$.

Proposition

The function

$$\zeta(z) := \frac{1}{\det(I - zA)} = \sum_{h \in \mathcal{H}} z^{|h|}$$

is the hikes generating function.

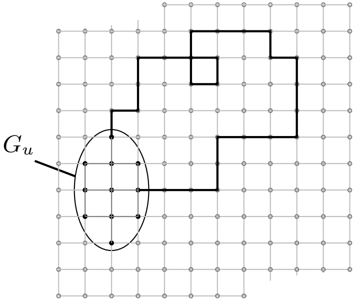
- $\zeta(.)$ is the zeta function on $\mathcal{H}$
- $z \mapsto \det(I - zA)$ is the Möbius function

Remark

The closed walks on G are elements of $\mathcal{H}$ with a unique right simple cycle.

Definition

An excursion on u in a graph G is a walk that starts and ends in u but avoids u in between.



Generating function:

$E_u(z) = \ ?$

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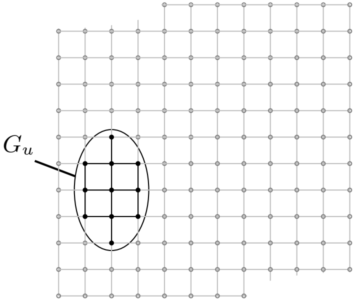

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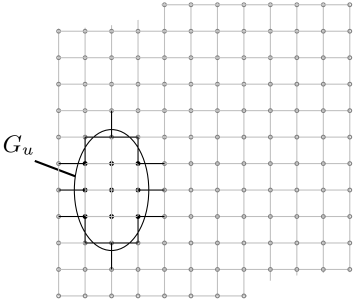

Generating function:

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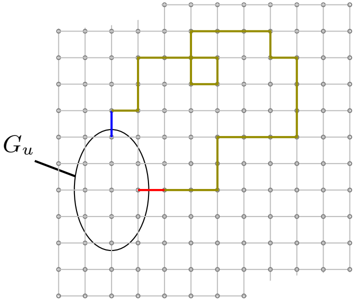

Generating function:

$$E_u(z) = zA_{uu} + z^2 A_{u\overline{u}} A_{\overline{u}u} + \sum_{k\geq 3} z^k A_{u\overline{u}} \, A_{\overline{u}\overline{u}}^{k-2} \, A_{\overline{u}u}$$

Generating function of the excursions on u

$$E_u(z) = zA_{uu} + z^2 A_{u\overline{u}}(I - zA_{\overline{u}\overline{u}})^{-1} A_{\overline{u}u}$$

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$$\iff$$

$$I - E_u(z) = (I - zA)_{uu} - z^2 A_{u\overline{u}}\big((I - zA)_{\overline{u}\overline{u}}\big)^{-1} A_{\overline{u}u}$$

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Unique factorisation of a walk from u to u as a product of excursions

$$I + E_u(z) + E_u(z)^2 + \dots = \big(I + zA + z^2 A^2 + \dots\big)_{uu}$$

Proposition

The function

$$\zeta_u(z) := \frac{1}{\det(I - E_u(z))}$$

is the generating function of hikes whose right simple cycles all intersect u.

Or

$$\zeta_u(z) = \det\left(\left((I - zA)^{-1}\right)_{uu}\right) = \mathbb{E}\left(\prod_{i \in u} \frac{1}{1 - zX_i}\right)$$

Combinatorical expression on the homogenous k-moments of $X_i, i \in u$.