Trace monoids, hike monoids and number theory

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Joint work with Pierre-Louis Giscard and Jean Fromentin

WACA, Calais

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2. Hikes properties and number theory
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   - Surjectivity

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Trace monoids

Definition

A *trace monoid* $\mathcal{M}$ is given by:

- a set S of generator,
- a set $\mathcal{I}$ of pairs of commuting generators called *independence relations*.

We denote $\mathcal{M} =< S \,|\, \mathcal{I} >$.

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We denote $\mathcal{M} =< S \,|\, \mathcal{I} >$.

Example: The set of natural integers greater than two with multiplication is the trace monoid with generators the primes and no independence relations: $(\mathbb{N} \setminus \{0,1\}\,, \times) =< \mathbb{P} \,|\, \mathbb{P}^2 >$.

Trace monoids

$$\text{Trace monoids} \iff \text{graphs}$$

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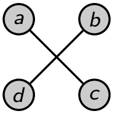
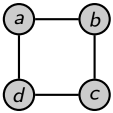
Example: $\mathcal{M} =< a, b, c, d \mid ac = ca, bd = db >$

Independence graph



Dependence graph



Cartier-Foata monoids

Let $\mathcal{G} = (V, E)$ be a digraph.

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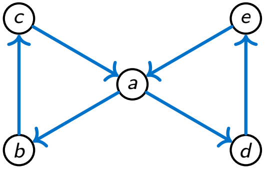

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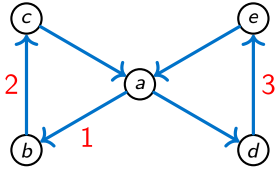

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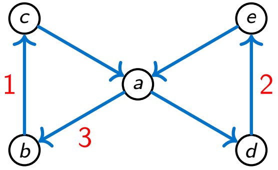

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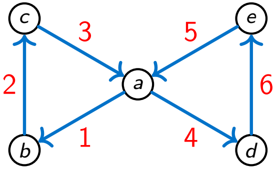

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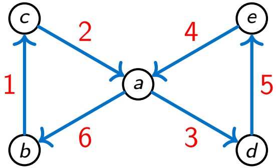

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Hike monoids

A *hike* of $\mathcal{G}$ is an element $w_{i_1 j_1} \cdots w_{i_n j_n} \in \mathcal{M}_{\mathcal{G}}$ such that for every $v \in V$

$$\#\{k \mid i_k = v\} = \#\{k \mid j_k = v\}.$$

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Proposition (Hike monoid)

This sub-monoid is isomorphic to the trace monoid with generators the induced cycles of $\mathcal{G}$ and with independence relations the pairs of disjoint cycles.

Hike monoids

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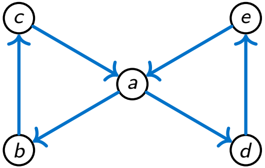

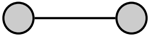

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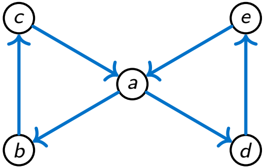
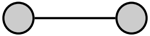
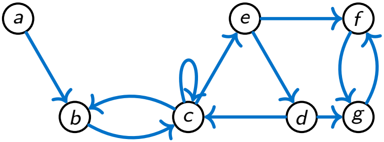
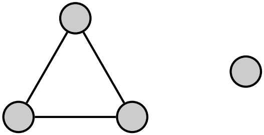




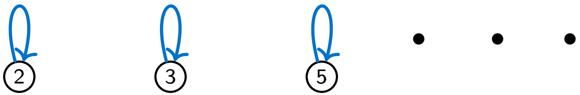




Hike monoids

Remark:

- $(\mathbb{N} \setminus \{0, 1\}, \times)$ is a hike monoid:



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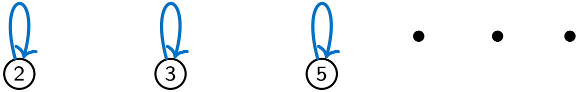
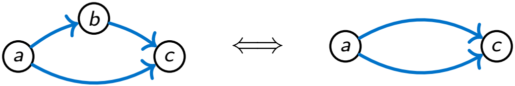


- 

$\rightarrow$ wloss multidigraphs

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$d, h \in \mathcal{H}$.
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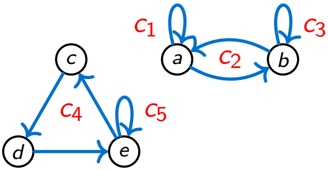

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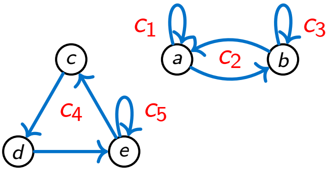

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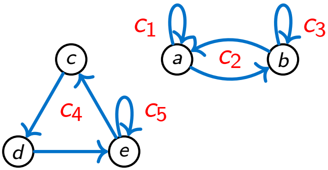

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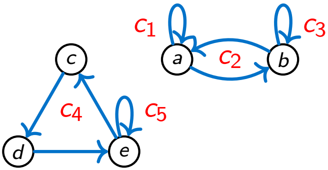

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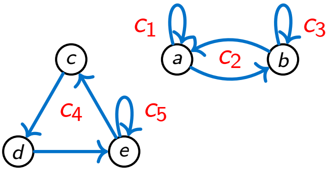

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Remark: Usual division on $\mathbb{N} \setminus \{0, 1\}$.

Hike incidence algebra

Hike incidence algebra: $\mathcal{F} = \mathcal{G} \to \mathbb{R}$ endowed with

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Proposition

$$\mathcal{S}f(s)\mathcal{S}g(s) = \mathcal{S}f * g(s)$$

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W: formal square matrix of size $\#V$ defined by:

$$W[i,j] = \begin{cases} 0 & \text{if } (i,j) \notin E \\ m_{i,j} w_{i,j} & \text{if } (i,j) \text{ appears } m_{i,j} \text{ times in } E \end{cases}$$

Möbius function

- δ the unit: $\delta(1) = 1$ and $\delta(h) = 0$ for $h \neq 1$,
- 1 the function constant equal to 1,
- μ the Möbius function: inverse of 1 for the Dirichlet convolution:

$$\mu * 1 = \delta,$$

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- $\Omega : h \mapsto$ number of primes in prime decomposition.

Proposition ([2])

$$\mu(h) = \begin{cases} 1 & \text{if } h = 1 \\ (-1)^{\Omega(h)} & \text{if } h \text{ is self-avoiding} \\ 0 & \text{else.} \end{cases}$$

In particular, on $\mathbb{N}$ μ coincides with the number theoretic Möbius function.

Möbius function

Proposition ([1])

$$\mathcal{S}\mu = \det(Id - e^{-s}W),$$
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Remark: Other determinantal expressions [3].

Walk von Mangoldt function [1]

The *walk von Mangoldt function* $\Lambda : \mathcal{H} \to \mathbb{N}$ is defined as the number of contiguous representations of a hike:

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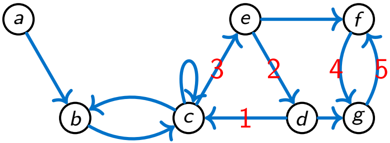

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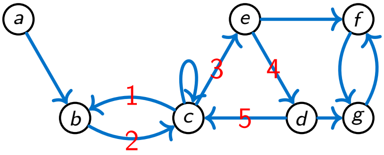

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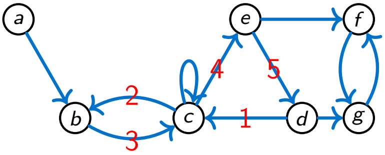

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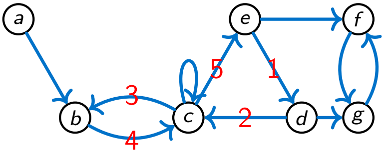

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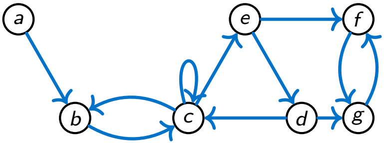

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On $\mathbb{N}$, $\Lambda(h) = \begin{cases} \ell(p) & \text{if } h = p^k \text{ with } p \text{ prime} \\ 0 & \text{else.} \end{cases}$

Walk von Mangoldt function [1]

Denote by ζ the series $\mathcal{S}1$.

Proposition ([1])

$$\mathcal{S}\Lambda(s) = -\frac{\zeta'(s)}{\zeta(s)},$$

$$\log \zeta(s) = \sum_{h \in \mathcal{H}} e^{-s\ell(h)} \frac{\Lambda(h)}{\ell(h)} h.$$

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Analogous to number theory, replacing length by logarithm.

Totally additive functions

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On $\mathbb{N}$, we recover the number theoretic version:

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- Injectivity: link between digraphs representing the same trace monoid ?
- Surjectivity: which trace monoid are representables ?

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Some transformations:

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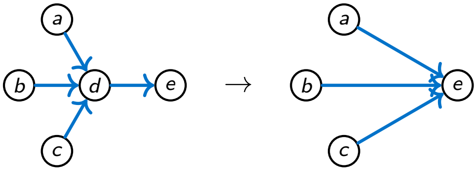
- Quotient by edges which are sole out/in going.



Injectivity

Some transformations:

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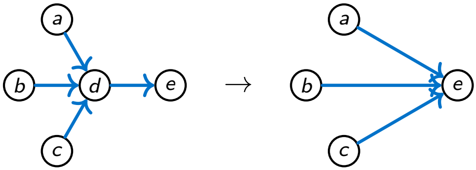
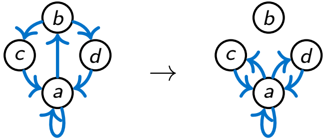


- If all cycles containing v also contain v', 'jump' v.



Injectivity

Observation: Every vertex in $\mathcal{G}$ correspond to a clique in the dependence graph of $\mathcal{H}(\mathcal{G})$.

Injectivity

Observation: Every vertex in $\mathcal{G}$ correspond to a clique in the dependence graph of $\mathcal{H}(\mathcal{G})$.

Conjecture: Let $\mathcal{G}$ be a (multi)digraph. Let $\mathcal{C}$ be a minimal number of cliques covering $\mathcal{H}(\mathcal{G})$. Then there exists a multidigraph $\mathcal{G}'$ with $\#\mathcal{C}$ vertices such that $\mathcal{H}(\mathcal{G}) = \mathcal{H}(\mathcal{G}')$.

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1 Trace and hikes monoids

2 Hikes properties and number theory
- Divisibility and incidence algebra
- Examples

3 Two problems
- Injectivity
- Surjectivity

First observations

$\mathcal{H}(\mathcal{G} \sqcup \mathcal{G}') = \mathcal{H}(\mathcal{G}) \sqcup \mathcal{H}(\mathcal{G}') \longrightarrow$ restrict to connected graphs.

First number of representable connected graphs (up to isomorphism): $1, 1, 2, 5, 15, 58, 265$. Not in oeis.

Some patterns

Algorithm

Let $\mathcal{M}$ be a trace monoid.

- Let $\mathcal{C}$ be a minimal set of cliques covering the dependence monoid of $\mathcal{M}$.
- Let P be the poset on $\left\{\bigcap_{k \in S} k \mid S \subseteq \mathcal{C}\right\}$ ordered by inclusion.
- Let $f : P \to \mathbb{N}$ be the map defined by $f(K) = \#K \setminus \bigcup_{K' < K} K'$.
- For every pair intersecting pair of cliques $k, k' \in \mathcal{C}$, let $m_{kk'}$ and $m_{k'k}$ be formal variables.
- Let $\mathscr{S}$ be the following system:

$$\forall K \in P, \sum_{\substack{(k_1,\ldots,k_n) \in \mathcal{C}^n \\ \cap k_i = K}} m_{k_1 k_2} \cdots m_{k_{l-1} k_l} m_{k_l k_1} = f(K)$$

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Algorithm

Conjecture: The system $\mathscr{S}$ has a solution in $\mathbb{N}$ if and only if $\mathcal{M}$ is representable. Furthermore one of the solutions is such that the multidigraph $\mathcal{G}$ over $\mathcal{C}$ and where the edge (k, k') appears $m_{kk'}$ times is such that $\mathcal{H}(\mathcal{G}) = \mathcal{M}$

Other trails

- New idea of algorithm by Jean Fromentin.
- Talk with Xiaolin Zeng.

References

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[2] P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Math., 85 (1969).

[3] C. Choffrut and M. Goldwurm, Determinants and Mobius functions in trace monoids, Discrete Math., 194 (1999), pp. 239–247.

Hike incidence algebra

Proof: $f, g \in \mathcal{F}$

$$\mathcal{S}f(s)\mathcal{S}g(s) = \sum_{h\in\mathcal{H}} e^{-s\ell(h)} g(h)h \sum_{h\in\mathcal{H}} e^{-s\ell(h)} g(h)h$$

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&= \sum_{h,h'\in\mathcal{H}} e^{-s\ell(hh')} f(h)g(h')hh' \\
&= \sum_{h\in\mathcal{H}} e^{-s\ell(h)} \left(\sum_{d|h} f(d)g\left(\frac{h}{d}\right)\right) h
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&= \sum_{h,h'\in\mathcal{H}} e^{-s\ell(h)} f(h) h e^{-s\ell(h')} g(h')h' \\
&= \sum_{h,h'\in\mathcal{H}} e^{-s\ell(hh')} f(h) g(h') hh' \\
&= \sum_{h\in\mathcal{H}} e^{-s\ell(h)} \left(\sum_{d|h} f(d) g\left(\frac{h}{d}\right) \right) h \\
&= \sum_{h\in\mathcal{H}} e^{-s\ell(h)} f * g(h) h
\end{aligned}
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Totally multiplicative functions

A function $f \in \mathcal{F}$ is *totally additive* if $f(hh') = f(h)f(h'), \forall h, h' \in \mathcal{H}$.

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Proposition ([1])

The inverse of a totally multiplicative function f for the Dirichlet convolution is given by

$$f^{-1} = \mu f.$$

In term of series: $\sum_{h\in\mathcal{H}} e^{-s\ell(h)} f(h)h = \frac{1}{\sum_{h\in\mathcal{H}} e^{-s\ell(h)}\mu(h)f(h)h}$,
Generalization of MacMahon's master theorem.

Bidirected digraphs

A bidirected digraphs $\mathcal{G}$ is a digraph without loops and such that $(i,j) \in \mathcal{G} \iff (j,i) \in \mathcal{G}$.

Theorem ([1])

The map $\mathcal{G} \to \mathcal{H}(\mathcal{G})$ restricted to bidirected digraphs is injective up to isomorphism except for K_3 and $K_{1,5}$.