A co-preLie structure from chronological loop erasure in graph walks

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Abstract

We show that the chronological removal of cycles from a walk on a graph, known as Lawler's loop-erasing procedure, generates a preLie co-algebra on the vector space spanned by the walks. In addition, we prove that the tensor and symmetric algebras of graph walks are Hopf algebras, provide their antipodes explicitly and recover the preLie co-algebra from a brace coalgebra on the tensor algebra of graph walks. Finally we exhibit sub-Hopf algebras associated to particular types of walks.

Keywords: Graphs, walks, cycles, coproduct, co-preLie co-algebra, Hopf algebra

Introduction

Graphs and walks are ubiquitous objects in combinatorics, discrete mathematics and beyond: they appear throughout linear algebra, differential calculus and have found widespread applications in physics, engineering and biology. Yet, while graph theory is being developed, less attention has been devoted to the walks themselves, a walk being a contiguous succession of directed edges on a graph. In particular the algebraic structures associated to walks have not, to the best of our knowledge, been fully explored. We may here refer the reader to quivers and path algebras and hike monoids [4]. The goal of the present work is to exhibit a co-preLie structure naturally associated to walks on graphs (simple graphs, multi-graphs, digraphs and hypergraphs). The structure arises from a simple procedure, now known as *Lawler's loop erasing* [7], first conceived in the context of percolation theory to randomly generate simple paths–walks where all vertices are distinct–from a sample of random walks. The procedure consists of a chronological removal of cycles (called loops in Lawler's original work) as one walks along on the graph: consider for instance the complete graph K_4 on 4 vertices and label these vertices with integers 1 through 4. Walking along the path $1 \to 2 \to 1 \to 3 \to 4 \to 3 \to 1 \to 3$ on the graph and removing cycles whenever they appear, we are left with the simple path $1 \to 3$ after having successively 'erased' the cycles $1 \to 2 \to 1$, then $3 \to 4 \to 3$ and finally $1 \to 3 \to 1$. Note how $1 \to 3 \to 1$ does not appear contiguously in the original walk. Once terminated, Lawler's loop-erasing has eliminated a set of cycles, all of whose internal vertices are distinct, leaving a possibly trivial walk-skeleton behind. If the initial walk was itself a cycle, this skeleton is the empty walk on the initial vertex (also called length-0 walk) and otherwise it is a simple path (also called self-avoiding path). Remark that because the loop-removal occurs in a chronological fashion, Lawler's process is strongly non-Markovian: complete knowledge of all the past steps of a walk is required to decide the current and future erased sections at any point of the walk.

We show below that this intuitive process is naturally associated with a co-preLie coproduct. In addition, slightly relaxing the chronological constraints by allowing simultaneous erasures under some

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compatibility conditions leads to Hopf algebra structures on the tensor and symmetric algebras of graph walks.

The article is organized as follows. In Section §1 we begin with basic notations and definitions concerning walks, graphs and Lawler's loop erasing procedure. In §2 we describe the chronological structure that walks acquire from Lawler's process and use this structure to define the admissible cuts of a walk. We show in particular that this notion is well defined in the sense that in spite of the strong chronological constraints created by Lawler's process, cutting out admissible cuts does not alter the other cuts admissibility. This leads in §3 to the definition of a co-product on walks which we show to be co-preLie. Then, in §4, considering a wider set of simultaneously admissible cuts, called extended admissible cuts, we construct a co-associative co-product on the tensor and symmetric algebras generated by the vector space of walks on a graph. We then prove an explicit formula for the antipode maps in the so-obtained Hopf algebras. In §5 we construct a brace coalgebra and a codendriform bialgebra on the tensor algebra generated by graph walks and use these to recover the preLie structure as a corollary of the Hopf algebra of the preceding section. Finally in §6 we exhibit Hopf subalgebras associated to certain types of walks, the cacti, ladders and corollas.

In a subsequent work inspired by previous combinatorial results [5], we will show that Lawler's process is also naturally associated with a non-associative permutative product, known as nesting [5], which satisfies the Livernet compatibility condition [8] with the co-preLie co-product defined here. This will provide the very first concrete example of the NAP - co-preLie structure in a 'living' context. This construction appears to be of paramount importance given the pervasive use of graph-walks in mathematics and mathematical-physics. In particular, we will show that this leads to a useful bridge between formal sums over infinite families of walks and branched continued fractions.

1. Notations and definitions

Throughout this work we use calligraphic font e.g., $\mathcal{W}$, $\mathcal{V}$, $\mathcal{T}$ for vector spaces and algebras, and upper-case e.g., V, E, LES for sets.

1.1. Notations for graphs and rooted walks

While we begin by recalling standard definitions for graphs, we introduce somewhat less common concepts for walks, of which we advise the reader to take special notice.

A *graph* $G = (V, E)$ is a countable set of vertices V and a countable set E of distinct paired vertices, called edges, denoted $\{i, j\}$, $i, j \in V$. A *digraph* $G = (V, E)$ is a finite set of vertices V and a finite set $E \subseteq V^2$ of *directed edges* (or *arcs*), denoted (i, j) for the arc from i to j. A *directed multigraph* (or *multidigraph*) is defined the same way as a digraph, except that E is a multiset. An edge of E is then denoted $(i, j)_k$, the integer k specifying which edge from i to j we consider. In the present work we always assume that G is non-empty.

A *rooted walk*, or rooted path, of length ℓ from vertex i to vertex j on a multi directed graph G is a contiguous sequence of ℓ arcs starting from i and ending in j, e.g. $\omega = (i, i_1)_{k_1}(i_1, i_2)_{k_2} \cdots (i_{\ell-1}, j)_{k_\ell}$ (a sequence of arcs is said to be contiguous if each arc but the first one starts where the previous ended). The rooted walk ω is *open* if $i \neq j$ and *closed* otherwise, in which case it is also called *rooted cycle*. Since we only consider rooted walks in this work, we shorten this terminology to *walks*. On digraphs we may unambiguously represent walks simply as ordered sequences of vertices $\omega = w_0 w_1 \cdots w_{\ell-1} w_\ell$. The walk $\omega = w_0$ of length 0 is called the *trivial* walk on vertex w_0, it is both open and closed. The set of all walks of length greater or equal to one on a graph G is denoted $\mathcal{W}(G)$.

Consider a walk $\omega = w_0 \dots w_\ell$. A *subwalk* of a walk $\omega = w_0 \cdots w_\ell$ is any walk $w_k \cdots w_{k'}$ where $0 \leq k \leq k' \leq \ell$. If $k \neq k'$ and $w_k = w_{k'}$, we designate by $\omega^{k,k'} := w_k w_{k+1} \dots w_{k'}$ the *closed subwalk* of ω with root w_k. In a complementary way, we define the *remainder* section $\omega_{k,k'} := w_0 \dots w_k w_{k'+1} \dots w_\ell$ to be what remains of ω after removal of the section $\omega^{k,k'}$. Note, for convenience we denote $\omega^{l,l'}_{k,k'}$ for $(\omega_{k,k'})^{l,l'}$, the section $w_l \dots w_{l'}$ erased from the remainder $\omega_{k,k'} = w_0 \dots w_k w_{k'+1} \dots w_\ell$. This means in particular that in $\omega^{l,l'}_{k,k'}$, integers k, k', l and l' all refer to indices from ω.

A rooted walk in which all vertices are distinct is said to be a *simple path* or *self-avoiding walk*. The set of all such walks on a digraph G is denoted $\Pi(G)$. Similarly, a rooted cycle $(i_0, i_1)_{k_1}(i_1, i_2)_{k_2} \cdots (i_{\ell-1}, i_0)_{k_\ell}$ of non-zero length for which all vertices i_t are distinct is said to be a *simple cycle* or *self-avoiding polygon*. Note that a self-loop $(i, i)_k$ is considered a rooted simple cycle of length one. The set of all simple cycles on G is $\Gamma(G)$. For G any (directed multi)graph, to ease the notation, we also denote by $\mathcal{W}(G)$ the $\mathbb{K}$-vector space spanned by all walks of length greater or equal to one on G, $\mathbb{K}$ being a field of characteristic 0. For a walk $\omega \in \mathcal{W}(G)$, we designate by $V(\omega)$ the support of ω, that is the set of *distinct* vertices visited by ω; and by $E(\omega)$ the *multiset* of directed edges visited by ω.

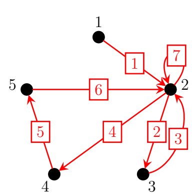
1.2. Definitions for loop-erasure

As stated in the introduction, Lawler's loop-erasing procedure consists in erasing all cycles from a walk ω in the *chronological* order in which they appear. Formally, it is a selection-quotient process which transforms a walk into its self-avoiding skeleton. To construct the algebraic structures associated with Lawler's procedure we must not only consider its end product but also what it produces during its intermediary stages and what it removes from the walk, in its original context:

Definition 1 (Loop-erased sections). Let G be a digraph and consider $\omega = w_0 \dots w_\ell \in \mathcal{W}(G)$. The set $\mathrm{LES}(\omega)$ of loop-erased sections is the set of all *closed subwalks* of ω *erased* by Lawler's procedure.

Remark 1. Let $\omega \in \mathcal{W}(G)$, then the set $\mathrm{LES}(\omega)$ is a subset of $\mathcal{W}(G)$.

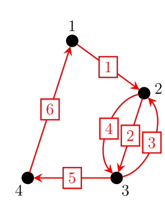
Example 1. On the complete graph K_5 on 5 vertices (including self-loops), consider the walk $\omega = 12324522$,

$\omega =$ 

In this illustration, the integers in boxes in the middle of edges give these edges' order of traversal while vertices are labeled by black integers next to them. The simple cycles erased by Lawler's procedure are $\omega^{1,3} = 232$, $\omega^{3,6} = 2452$ and $\omega^{6,7} = 22$ and the set of erased closed subwalks of ω is therefore,

$$\begin{aligned}\mathrm{LES}(12324522) =& \{\omega^{1,3}, \omega^{3,6}, \omega^{1,6}, \omega^{6,7}, \omega^{3,7}, \omega^{1,7}\} \\ =& \{232, 2452, 232452, 22, 24522, 2324522\}.\end{aligned}$$

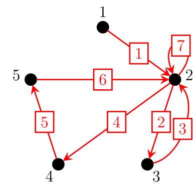
Remark 2. The requirement that the closed subwalks of $\mathrm{LES}(\omega)$ be constructed solely from *erased* sections is crucial. For example, in

$\omega = 1232341 =$ 

we have $\mathrm{LES}(\omega) = \{\omega^{1,3}, \omega\} = \{232, 1232341\}$. In particular $323 \notin \mathrm{LES}(\omega)$ because 323 was not erased at once by Lawler's procedure. Including it would violate the chronological condition innate in Lawler's process, as erasing 323 from ω would imply having overlooked the cycle 232 which was closed *prior* to 323. We formalize this observation with the notion of loop-erased walks:

Definition 2 (Loop-erased walks). Let G be a digraph and $\omega = w_0 \cdots w_\ell \in \mathcal{W}(G)$ of length ℓ. For $0 \le k \le \ell$, we designate $\mathsf{Lew}_k(\omega)$, called loop-erased walk ω at step k, to be what is left of ω after its first k steps while performing Lawler's procedure. We simply write $\mathsf{Lew}(\omega) \equiv \mathsf{Lew}_\ell(\omega)$ for the final, self-avoiding skeleton of ω left at the end of the procedure. Note that $\mathsf{Lew}_k(\omega)$ is not a set but, by construction, a walk.

Example 2. Consider the walk,

$\omega = 12324522 =$ 

Then $\mathsf{Lew}_0(\omega) = 1$, $\mathsf{Lew}_1(\omega) = 12$, $\mathsf{Lew}_2(\omega) = 123$, $\mathsf{Lew}_3(\omega) = 12$, $\mathsf{Lew}_4(\omega) = 124$, $\mathsf{Lew}_5(\omega) = 1245$, $\mathsf{Lew}_6(\omega) = 12$, $\mathsf{Lew}_7(\omega) = \mathsf{Lew}(\omega) = 12$.

Remark 3. Let $\omega = w_0 \cdots w_\ell \in \mathcal{W}(G)$. Then $\mathsf{Lew}(\omega)$ is the trivial walk on vertex w_0 if and only if ω is closed.

By the definitions of $\mathrm{LES}(\omega)$ and $\mathsf{Lew}(\omega)$ we obtain what was remarked above, namely that loop-erased sections may not straddle over one-another, a consequence of their step-by-step erasure in chronological order:

Lemma 1. *Let G be a digraph and $\omega = w_0 \dots w_\ell \in \mathcal{W}(G)$. Then $\omega^{k,k'} \in \mathrm{LES}(\omega)$ if and only if there does not exist a pair of integers $0 \le l < k < l' < k' \le \ell$ with $w_k = w_{k'} \neq w_l = w_{l'}$ and $\omega^{l,l'} \in \mathrm{LES}(\omega)$*

Before we prove the lemma, we remark that the notion of loop-erased walks allows for an alternative but equivalent definition of that of loop-erased section:

Remark 4 (A recursive procedure for constructing LES(ω)). Let G be a digraph and consider $\omega = w_0 \dots w_\ell \in \mathcal{W}(G)$. The set $\mathrm{LES}(\omega)$ of loop-erased sections of $\omega = w_0 \cdots w_\ell$ is constructed recursively as follows. Initialize with $\mathrm{LES}(\omega) = \emptyset$. Then for $k \in \{1, \dots, \ell - 1\}$, if $w_{k+1} \in V(\mathsf{Lew}_k(\omega))$ denote k', the greatest integer such that $0 \le k' \le k$ and $w_{k'} = w_{k+1}$. If k' exists, then:

1. add the closed walk $\omega^{k',k+1} = w_{k'} \dots w_{k+1}$ to the set $\mathrm{LES}(\omega)$ of loop-erased sections of ω;

2. if there exists $\omega^{k'',k'}$ a loop-erased section of ω, add the closed walk $\omega^{k'',k+1} = w_{k''} \dots w_{k+1}$ to the set $\mathrm{LES}(\omega)$ of loop-erased sections of ω as well.

While equivalent to Definition 1, the above formulation is more formal in flavor and recursive in nature, thus better suited to algorithm designs and easier to wield in proofs.

Example 3. Consider again the walk of Example 2, $\omega = 12324522$, and let us illustrate the construction of $\mathrm{LES}(\omega)$, the set of loop-erased sections of ω, as expounded in Remark 4.

1. At the beginning, $\mathrm{LES}(\omega) = \emptyset$.

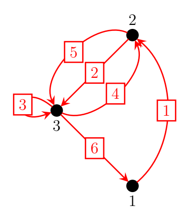
2. For $k = 0, 1$, $\mathrm{LES}(\omega)$ remains empty but as $2 \in V(\mathsf{Lew}_2(\omega))$ section $\omega^{1,3} = 232$ is erased, added to the set of loop-erased sections $\mathrm{LES}(\omega) = \{\omega^{1,3}\} = \{232\}$, while now $\mathsf{Lew}_3(\omega) = 12$.

3. Nothing is added to the set of loop-erased sections until $k = 6$, at which point observing that $2 \in V(\mathsf{Lew}_5(\sigma))$ section $\omega^{3,6} = 23452$ is to be erased and must be added to $\mathrm{LES}(\omega)$. Since furthermore $\omega^{1,3} \in \mathrm{LES}(\omega)$, by point 2. of Remark 4, $\omega^{1,6}$ ought to be added as well. Then we get
$$\mathrm{LES}(\omega) = \{\omega^{1,3}, \omega^{3,6}, \omega^{1,6}\} = \{232, 2452, 232452\}.$$
4. At the last step something similar happens with the erasure of $\omega^{6,7} = 22$ which leads by point 2. of Remark 4 to
$$\begin{aligned}\mathrm{LES}(\omega) =& \{\omega^{1,3}, \omega^{3,6}, \omega^{1,6}, \omega^{6,7}, \omega^{3,7}, \omega^{1,7}\}\\ =& \{232, 2452, 232452, 22, 24522, 2324522\}.\end{aligned}$$

Proof of Lemma 1. Assuming that $\omega^{k,k'}, \omega^{l,l'} \in \mathrm{LES}(\omega)$, suppose that both sections nonetheless straddle over one-another. We may choose without loss of generality (wlog) that $k < l < k' < l'$. In particular, there is no earlier step $m < k$ with $w_m = w_l$ since otherwise we would effectively be in the straddling situation where $l < k$. Then at step $l' - 1 \geq k'$ of the walk, vertex $w_l \notin V(\mathsf{Lew}_{l'-1})(w)$ since at this point $\omega^{k,k'}$ has already been erased and so by Remark 4, $\omega^{l,l'} \notin \mathrm{LES}(\omega)$, a contradiction. Conversely, if the pair of integers l and l' as stated does not exists, then $\omega^{k,k'}$ defines a closed subwalk of ω erased by Lawler's procedure, hence in $\mathrm{LES}(\omega)$. □

2. The chronological structure of walks

From a walk ω, Lawler's process, once terminated, produces a set of erased simple cycles and one self-avoiding skeleton (possibly trivial). It is therefore natural to seek a co-product which to the walk ω would associate a sum over erased sections $\omega^{k,k'}$ and associated remainders $\omega_{k,k'}$, so that ω could be obtained back from these through grafting of the former onto the latter. The 'grafting' product appropriate to that end, known as *nesting*, was first identified thanks to purely combinatorial considerations [5] and is permutative non-associative reflecting Lawler's process' chronological constraints. It is difficult to maintain any form of compatibility with nesting via such an indiscriminate procedure as cutting out all loop-erased sections however, as not all pairs $(\omega^{k,k'}, \omega_{k,k'})$ can be consistently grafted back to form the original walk; and when grafting is possible, it may be so in more than one way. These problems arise from certain ladders and all corollas, respectively. Consider first an instance of the former,

$\omega = 1233231 =$ 

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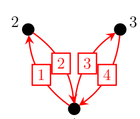
which is a ladder in the sense that the self-loop 33 is attached 'on top of' cycle 232, itself attached to the 'base' triangle 1231. Here $\omega^{2,3} = 33$ is a valid loop-erased section of ω, yet can be grafted back onto $\omega_{2,3}$ in two distinct ways: one producing ω and the other yielding the walk $\omega' = 1232331$. Remark how in ω', the self-loop 33 occurs one level below its original location in ω since it is now attached directly to the 'base' triangle 1231. Algebraically such instances correspond to cases where the nesting product fails to be associative. Second, for the issue with corollas, i.e. bouquets of closed walks with the same root, consider e.g.

$\omega = 12131 =$ 

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Here both 121, $131 \in \mathrm{LES}(\omega)$; yet cutting e.g. $\omega^{0,2} = 121$ and grafting it back onto $\omega_{0,2} = 131$ either gives back the walk $\omega = 12131$ or the completely different one $\omega' = 13121$. Algebraically, these instances translate into cases where the nesting product fails to be commutative.

2.1. Admissible cuts

To resolve the difficulties mentioned above, which become extensive when taken together in arbitrary long walks, we must refine the set of loop-erased sections that can be cut out of the original walk by the co-product. Here, as earlier, the major hurdle is due to the chronological constraints inherent to Lawler's process. Because of this, special attention must be paid to erased sections that appear within longer erased sections, the latter providing the temporal context of the former:

Definition 3 (Temporal context of an erased section). Let G be a digraph, $\omega \in \mathcal{W}(G)$ and $\omega^{k,k'} \in \mathrm{LES}(\omega)$. We denote $\mathrm{LES}(\omega)^{<}_{k,k'} \subset \mathrm{LES}(\omega)$ the subset of loop-erased sections $\omega^{l,l'}$ which *strictly* include $\omega^{k,k'}$ as left subwalk, i.e. $l \leq k < k' < l'$. Because we require $k' < l'$ strictly, $\mathrm{LES}(\omega)^{<}_{k,k'}$ may be empty. Otherwise, we denote $\omega^{\min}_{k,k'}$ the smallest element of $\mathrm{LES}(\omega)^{<}_{k,k'}$ for inclusion.

By construction, if $\omega^{\min}_{k,k'}$ exists, it is the tightest erased section which encompasses $\omega^{k,k'}$ entirely. It provides the relevant temporal context for $\omega^{k,k'}$ since anything outside of $\omega^{\min}_{k,k'}$ creates no further chronological constraints on $\omega^{k,k'}$ beyond those on $\omega^{\min}_{k,k'}$. This is because vertices appearing in the loop-erased walk at the start of $\omega^{\min}_{k,k'}$ cannot appear again inside of it by Lemma 1, so are necessarily avoided by $\omega^{k,k'}$. Hence, any additional constraint that Lawler's process imposes on $\omega^{k,k'}$ as compared to $\omega^{\min}_{k,k'}$ arise solely from within $\omega^{\min}_{k,k'}$.

Example 4. Let $\omega = 12324522$ be the walk of Examples 1–3 and consider its loop-erased section $\omega^{1,3}$. Since $\mathrm{LES}(\omega) = \{\omega^{1,3}, \omega^{3,6}, \omega^{1,6}, \omega^{6,7}, \omega^{3,7}, \omega^{1,7}\}$ (see Example 3), then $\mathrm{LES}(\omega)^{<}_{1,3} = \{\omega^{1,6}, \omega^{1,7}\}$. Indeed, both section $\omega^{1,6}$ and $\omega^{1,7}$ strictly contain $\omega^{2,4}$. Furthermore, the smallest of these by inclusion is $\omega^{\min}_{1,3} = \omega^{1,6}$, i.e. $\omega^{1,6}$ is the shortest loop-erased section strictly containing $\omega^{1,3}$. At the opposite, there is no loop-erased section strictly containing $\omega^{3,7} \in \mathrm{LES}(\omega)$, that is $\mathrm{LES}(\omega)^{<}_{3,7} = \emptyset$ and $\omega^{\min}_{3,7}$ does not exist.

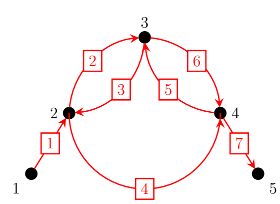
We can now control the loop-erased sections that a co-product may extract by admitting only those cuts which are corollas within their relevant temporal context and only if those cuts are contiguous subwalks including the last petals of the corolla:

Definition 4 (Admissible cuts). Let G be a digraph and $\omega = w_1 \dots w_\ell \in \mathcal{W}(G)$. A non-empty loop-erased section $\omega^{k,k'} := w_k w_{k+1} \dots w_{k'} \in \mathrm{LES}(\omega)$ is an *admissible cut* of ω when $\omega^{k,k'} \neq \omega$ and either $\omega^{l,l'} := \omega^{\min}_{k,k'}$ does not exist or w_k does not appear in $w_{k'+1} \cdots w_{l'}$. The set of admissible cuts of ω is denoted $\mathrm{AdC}(\omega)$.

Remark 5. Consider ω a walk and $\omega^{k,k'} \in \mathrm{LES}(\omega)$ a loop erased section from it. The condition that $\omega^{l,l'} := \omega^{\min}_{k,k'}$ either does not exist or w_k does not appear in $w_{k'+1} \cdots w_{l'}$ implies that admissible cuts can only be made right to left in the walk, that is from the latest to the earliest, in *reverse chronological order.*

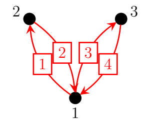
Example 5. In the complete graph K_5, consider the walk

$$\omega = 12324345 =$$



The loop-erased sections $\omega^{1,3} = 232 \in \mathrm{LES}(\omega)$ and $\omega^{4,6} = 434 \in \mathrm{LES}(\omega)$ are both admissible cuts of ω. At the opposite, $\omega^{2,5} = 3243 \notin \mathrm{LES}(\omega)$ and so is not an admissible cut.

Example 6. In the walk

$$\omega = 12131 =$$ 

subwalk $\omega^{2,4} \in \mathrm{LES}(\omega)$ is an admissible cut of ω, while $\omega^{0,2} = 121 \in \mathrm{LES}(\omega)$ is not admissible because vertex 1 is visited again by $\omega^{\min}_{0,2}$ after completion of $\omega^{0,2}$.

The notion of admissible cut is well defined because the property of being admissible does not depend on the order in which admissible cuts are considered and removed from the original walk. In particular, if a loop-erased section is an admissible cut of an admissible cut of a walk or of its remainder, then it is an admissible of that walk and vice-versa. This is significant because it indicates that, in spite of the strong chronological constraints created by Lawler's process, cutting out admissible cuts does not alter the other cuts relevant temporal context and thence, their admissibility:

Proposition 2. *Let G be a digraph and $\omega \in \mathcal{W}(G)$.*

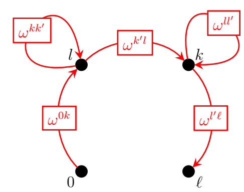
Case 1. *If $k < k' < l < l'$ or $l < l' < k < k'$ then,*

$$\omega^{k,k'} \in \mathrm{AdC}(\omega) \text{ and } \omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'}) \iff \omega^{l,l'} \in \mathrm{AdC}(\omega) \text{ and } \omega^{k,k'} \in \mathrm{AdC}(\omega_{l,l'}).$$

Case 2. *If $k < l < l' \le k'$ then,*

$$\omega^{k,k'} \in \mathrm{AdC}(\omega) \text{ and } \omega^{l,l'} \in \mathrm{AdC}(\omega^{k,k'}) \iff \omega^{l,l'} \in \mathrm{AdC}(\omega) \text{ and } \omega^{k,k'}_{l,l'} \in \mathrm{AdC}(\omega_{l,l'}).$$

Proof. **Case 1.** We assume $k < k' < l < l'$ without loss of generality (wlog), pictorially this is the situation where

$$\omega = w_0 \dots w_k \dots w_{k'} \dots w_l \dots w_{l'} \dots w_\ell =$$ 

Suppose that $\omega^{k,k'} \in \mathrm{AdC}(\omega)$ and $\omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'})$, we first establish that $\omega^{l,l'} \in \mathrm{AdC}(\omega)$.

Given that $\omega^{k,k'} \in \mathrm{LES}(\omega)$, a closed subwalk is erased from ω if and only if it is either erased from inside of the $\omega^{k,k'}$ section or from outside of it, i.e. $\omega_{k,k'}$. This is because, by Lemma 1, erased sections cannot straddle over one-another owing to their step-by-step erasure in chronological order. Here, $\omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'})$ and since $\omega_{k,k'} \in \mathrm{LES}(\omega)$, then $\omega^{l,l'}$ is an erased closed subwalk of within an erased closed subwalk of ω. This indicates that $\omega^{l,l'} \in \mathrm{LES}(\omega)$.

To show that $\omega^{l,l'}$ is an admissible loop-erased section of ω, consider the set $\mathrm{LES}(\omega)^{<}_{l,l'}$ of loop-erased sections of ω which strictly include $\omega^{l,l'}$ as subwalk. If $\omega^{\min}_{l,l'}$ does not exist, then $\omega^{l,l'}$ is admissible, $\omega^{l,l'} \in \mathrm{AdC}(\omega)$. Suppose instead that $\omega^{m,m'} := \omega^{\min}_{l,l'}$ exists and recall that $\omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'})$. This implies one of the two following possibilities:

i) $(\omega_{k,k'})^{\min}_{l,l'}$ does not exist then,

  (a) either $m \in \{1, \dots, k\} \cup \{k', \dots, l\}$ and thus $\omega^{m,m'}_{k,k'} \in \mathrm{LES}(\omega_{k,k'})^{<}_{l,l'} \neq \emptyset$ so its minimum exists, a contradiction;

  (b) or $m \in \{k+1, \dots, k'-1\}$ but then $\omega^{k,k'} \in \mathrm{AdC}(\omega)$ implies $\omega^{m,m'} \notin \mathrm{LES}(\omega)$, a contradiction.

ii) $\omega^{n,n'} := (\omega_{k,k'})^{\min}_{l,l'}$ exists, $n \le l < l' < n'$, and

(a) if $k < n < k'$ then $\omega^{k,k'} \in \mathrm{AdC}(\omega)$ implies $\omega^{n,n'} \notin \mathrm{LES}(\omega_{k,k'})$ a contradiction;

(b) if $n \geq k'$ then $\omega^{\min}_{l,l'} = (\omega_{k,k'})^{\min}_{l,l'}$ so $\omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'})$ implies $\omega^{l,l'} \in \mathrm{AdC}(\omega)$;

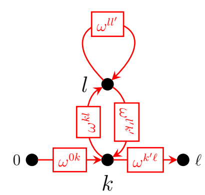
(c) if $n \leq k$ then $\omega^{n,n'} \in \mathrm{LES}(\omega)^{<}_{l,l'}$ i.e. either $\omega^{m,m'}$ is a subwalk of $\omega^{n,n'}$ or the two are the same and in both cases $\omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'})$ implies $\omega^{l,l'} \in \mathrm{AdC}(\omega)$.

This shows that $\omega^{l,l'} \in \mathrm{AdC}(\omega)$. Second, we establish that $\omega^{k,k'} \in \mathrm{AdC}(\omega_{l,l'})$.

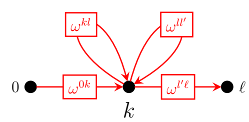
Since $\omega^{k,k'} \in \mathrm{AdC}(\omega)$ and given that $k' < l$ implies that $\omega^{k,k'}$ is an erased closed subwalk from outside of $\omega^{l,l'}$, then $\omega^{k,k'} \in \mathrm{LES}(\omega_{l,l'})$. Supposing that $\omega^{\min}_{k,k'}$ does not exist then $(\omega_{l,l'})^{\min}_{k,k'}$ does not exist either and $\omega^{k,k'} \in \mathrm{AdC}(\omega_{l,l'})$. Now suppose instead that $\omega^{\min}_{k,k'}$ exists, then $(\omega_{l,l'})^{\min}_{k,k'}$ is either identical to $\omega^{\min}_{k,k'}$ or is a subwalk of it. In both situations $\omega^{k,k'} \in \mathrm{AdC}(\omega)$ then entails that vertex $w_k = w'_k$ is not visited again after step k' in $\omega^{\min}_{k,k'}$ and its subwalks, hence $\omega^{k,k'} \in \mathrm{AdC}(\omega_{l,l'})$.

Conversely, assuming that $\omega^{l,l'} \in \mathrm{AdC}(\omega)$ and $\omega^{k,k'} \in \mathrm{AdC}(\omega_{l,l'})$ and proceeding as above we obtain that $\omega^{k,k'} \in \mathrm{AdC}(\omega)$ and $\omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'})$, which proves Case 1 of the Proposition.

Case 2. $k < l < l' \leq k'$, pictorially this is the situation where,

$$\omega = w_0 \dots w_k \dots w_l \dots w_{l'} \dots w_{k'} \dots w_\ell =$$  ,

or

$$\omega = w_0 \dots w_k \dots w_l \dots w_{k'=l'} \dots w_\ell =$$  .

We assume that $\omega^{l,l'} \in \mathrm{AdC}(\omega)$ and $\omega^{k,k'}_{l,l'} \in \mathrm{AdC}(\omega_{l,l'})$ and first establish that $\omega^{k,k'} \in \mathrm{AdC}(\omega)$.

Since $\omega^{k,k'}_{l,l'} \in \mathrm{LES}(\omega_{l,l'})$, by Lemma 1 there exists no pair of integers m, m' with $0 \leq m \leq k \leq m' \leq k' \leq \ell$ and $w_m = w_{m'} \in V(\mathsf{Lew}_k(\omega_{l,l'}))$. Additionally, $k < l < l' \leq k'$ implies that $\mathsf{Lew}_k(\omega_{l,l'}) = \mathsf{Lew}_k(\omega)$. Replacing the former with the latter gets $w_m \in V(\mathsf{Lew}_k(\omega))$ which entails that $\omega^{k,k'} \in \mathrm{LES}(\omega)$ by Lemma 1. It remains to verify that $\omega^{k,k'}$ is admissible. To this end remark that $(\omega_{l,l'})^{\min}_{k,k'}$ exists if and only if $\omega^{\min}_{k,k'}$ exists since $k < l < l' \leq k'$ and, by assumption, $\omega^{l,l'} \in \mathrm{AdC}(\omega)$. In this case $(\omega_{l,l'})^{\min}_{k,k'} = \omega^{\min}_{k,k'}$ so $\omega^{k,k'} \in \mathrm{AdC}(\omega)$.

Second we show that $\omega^{l,l'} \in \mathrm{AdC}(\omega^{k,k'})$. By assumption $\omega_{l,l'} \in \mathrm{AdC}(\omega)$, then $\omega^{l,l'}$ is a closed erased subwalk of ω and, by $k < l < l' \leq k'$, it is more precisely a subwalk of $\omega^{k,k'}$. Then $\omega^{l,l'} \in \mathrm{LES}(\omega^{k,k'})$.

To verify that the loop-erased section $\omega^{l,l'}$ is admissible in $\omega^{k,k'}$, observe that assumption $\omega^{k,k'}_{l,l'} \in \mathrm{AdC}(\omega_{l,l'})$ implies, depending on the situation:

i) if $l' < k'$ then $\omega^{k,k'} \in \mathrm{LES}(\omega^{k,k'})^{<}_{l,l'}$ and so $(\omega^{k,k'})^{\min}_{l,l'} = \omega^{\min}_{l,l'}$;

ii) if $l' = k'$ then $\mathrm{LES}(\omega^{k,k'})^{<}_{l,l'}$ is empty and $(\omega^{k,k'})^{\min}_{l,l'}$ does not exist.

Therefore in both situations $\omega^{l,l'}$ is an admissible cut of $\omega^{k,k'}$.

The converse results, namely proving that $\omega^{l,l'} \in \mathrm{AdC}(\omega)$ and $\omega^{k,k'}_{l,l'} \in \mathrm{AdC}(\omega_{l,l'})$ while assuming $\omega^{k,k'} \in \mathrm{AdC}(\omega)$ and $\omega^{l,l'} \in \mathrm{AdC}(\omega^{k,k'})$ are obtained completely similarly, yielding Case 2 of the Proposition. □

2.2. All walks are totally-ordered temporal trees

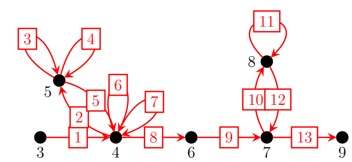
Lemma 1 and Proposition 2 strongly suggest that any walk on any graph is chronologically equivalent to a planar rooted tree where the root node is the self-avoiding skeleton of the walk and each non-root node stands for a simple cycle, see Theorem 4 below. In that tree, Lawler's procedure erases nodes from the leaves down to the root and operates on the branches from left to right (or more precisely along the direction given to time). That is, time totally orders the walk's tree structure. Formally, this translates into a total order on the set of admissible cuts of a walk:

Definition 5 (Time-ordering of the admissible cuts). Let G be a digraph and $\omega \in \mathcal{W}(G)$. Assuming that $\mathrm{AdC}(\omega) \neq \emptyset$ we define the relation $\leqslant_{\circlearrowright}$ on $\mathrm{AdC}(\omega)$ as follows:

$$\omega^{k,k'}, \omega^{l,l'} \in \mathrm{AdC}(\omega):\ \omega^{k,k'} \leqslant_{\circlearrowright} \omega^{l,l'} \iff l \leq k < k' \leq l' \text{ or } k < k' < l < l'.$$

Example 7. Consider the walk,

$$\omega = 34555444678879 =$$



and three of its admissible cuts, $\omega^{2,4} = 555$, $\omega^{3,4} = 55$ and $\omega^{10,11} = 88$. Then $\omega^{3,4} \leqslant_{\circlearrowright} \omega^{2,4} \leqslant_{\circlearrowright} \omega^{10,11}$.

Proposition 3. *Let G be a digraph, $\omega \in \mathcal{W}(G)$ and suppose that the set of admissible cuts of ω is non-empty. Then this set is totally ordered by $\leqslant_{\circlearrowright}$.*

Proof. By contradiction. Suppose that there exists two admissible cuts of ω, $\omega^{k,k'}$ and $\omega^{l,l'}$ such that $k < l'$ wlog, and which cannot be time-ordered. Then necessarily $k < l \leq k' \leq l'$. But since both $\omega^{k,k'}$ and $\omega^{l,l'}$ are admissible they are loop-erased sections of ω. But by Lemma 1 they may not straddle, that is we may not have $k < l \leq k' \leq l'$, a contradiction. Transitivity, reflexivity and anti-symmetry of $\leqslant_{\circlearrowright}$ follow immediately. □

Example 8. Consider again the walk of Example 7. It admissible cuts set is

$$\mathrm{AdC}(\omega) = \{\omega^{1,7}, \omega^{2,4}, \omega^{3,4}, \omega^{5,7}, \omega^{6,7}, \omega^{9,12}, \omega^{10,11}\},$$

and the total order on it is

$$\omega^{3,4} \leqslant_{\circlearrowright} \omega^{2,4} \leqslant_{\circlearrowright} \omega^{6,7} \leqslant_{\circlearrowright} \omega^{5,7} \leqslant_{\circlearrowright} \omega^{1,7} \leqslant_{\circlearrowright} \omega^{10,11} \leqslant_{\circlearrowright} \omega^{9,12}.$$

Theorem 4 (All walks are temporal-trees)**.** *Let G be a digraph and $\omega \in \mathcal{W}(G)$. Then ω has the temporal-structure of a tree $t(\omega)$ whose nodes are totally ordered by $\leqslant_{\circlearrowright}$ according to a depth-first order.*

Proof. To establish the theorem, we first map walks to cacti then cacti to trees:

Definition 6 (Cactus). Let G be a digraph. A walk $\omega = w_0 \dots w_\ell \in \mathcal{W}(G)$ is a cactus if and only if for any $0 \leq k < k' \leq \ell$, $w_k = w_{k'} \iff \omega^{k,k'} \in \mathrm{LES}(\omega)$. We denote $\mathcal{C}$ the set of all cacti on the complete graph $K_{\mathbb{N}}$ with $V(K_{\mathbb{N}}) = \mathbb{N}$ and by $\mathrm{Cact}(G)$ the vector space spanned by the cacti on G.

In a cactus all repeated vertices delimit valid loop-erased sections, which means that there cannot be patterns such as $\omega = 1\underline{2}1\underline{2}1$ as $212 \notin \mathrm{LES}(\omega)$. Intuitively a cactus is therefore a 'disentangled' walk, where every instance of repeated vertex is the root of a simple-cycle erased by Lawler's procedure. We may always map walks to cacti by defining,

$$C:\ \mathcal{W}(G) \to \mathcal{C}, \tag{1}$$
$$\omega = w_0 \cdots w_\ell \mapsto \kappa := C(\omega) = c_0 \cdots c_\ell,$$

where κ is the cactus defined as follows: $c_0 = w_0$ and, for any $k \in \{0, \dots, \ell - 1\}$,

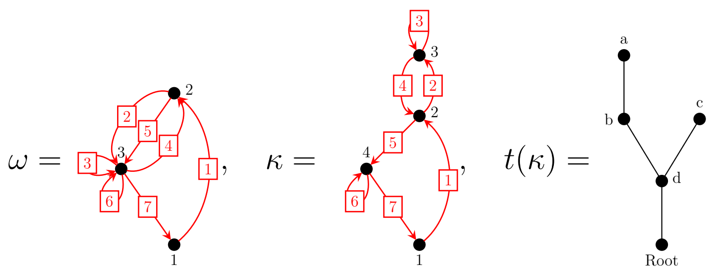
- if $\mathsf{Lew}_{k+1}(\omega) = \mathsf{Lew}_k(\omega)w_{k+1}$ then $c_{k+1} = \max(V(c_0 \dots c_k)) + 1$;
- else $c_{k+1} = c_l$ where $l = \max(i \in \{0, \dots, k\},\ w_l = w_{k+1})$.

In words, considering the loop-erased walk $\mathsf{Lew}_k(\omega)$ at step k, if vertex w_{k+1} reached at step $k+1$ is distinct from those of $\mathsf{Lew}_k(\omega)$ then c_{k+1} is a vertex with integer label given by the length of $\mathsf{Lew}_k(\omega)$ plus one (an expedient ensuring that we map distinct labels to distinct labels). If instead vertex w_{k+1} was visited at some step l prior to step $k+1$ in the loop-erased walk, that is $w_l \cdots w_{k+1}$ closes an erased-section, then c_{k+1} is given the same label as c_l. For example walk $\omega = 12121$ becomes $\kappa := C(\omega) = 12131$. Because the new labels are not labels of nodes of G, it may be that κ is not a valid walk on G but at least it is a walk on a complete graph $K_{\mathbb{N}}$, see §6.2 below. This is sufficient for our purpose: by definition ω and $\kappa := C(\omega)$ have the same length, κ is a cactus, and ω and κ share the same temporal structure,

$$\omega^{k,k'} \in \mathrm{AdC}(\omega) \iff \kappa^{k,k'} \in \mathrm{AdC}(\kappa). \tag{2}$$

Since any two simple cycles in cacti share at most one vertex (their roots), we define a tree $t(\kappa)$ from κ by drawing a tree-node t for every simple cycle σ of κ. Two nodes t and t' of the tree are connected if and only if the corresponding simple cycles σ and σ' share a vertex in κ. Finally we add a root node representing the (possibly trivial) self-avoiding skeleton of ω. The time-order $\leqslant_{\mathrm{clock}}$ now totally orders the nodes of $t(\kappa)$: for t, t' two nodes of $t(\kappa)$ corresponding to simple cycles σ and σ' of κ, $t \leqslant_{\mathrm{clock}} t' \iff \sigma$ is erased prior to σ in κ. This builds a reverse depth-first order on the nodes of $t(\kappa)$ with the top-left leaf of the tree corresponding to the first erased simple cycle, hence the smallest as per $\leqslant_{\mathrm{clock}}$. □

Example 9. As an example, consider the walk $\omega = 12332331$. Then $\kappa := C(\omega) = 12332441$ and the tree $\tau := t(\kappa)$ is



The simple cycles of κ are 1241, corresponding to node d of the tree; 232 (node b of the tree); 33 (node a) and 66 (node c). The root node of $t(\omega)$ stands for the trivial walk '1' on vertex 1, which is the self-avoiding skeleton of ω. The time-order on the tree nodes is $a \leqslant_{\mathrm{clock}} b \leqslant_{\mathrm{clock}} c \leqslant_{\mathrm{clock}} d \leqslant_{\mathrm{clock}}$ Root. This is closely related to the upper-left order $\geq_{tot}$ introduced by Foissy [2], which would lead to Root $< d < b < a < c$. In our situation node order increases from top to bottom and from left to right.

Although the tree $t(\kappa) = t(C(\omega))$ depends on the walk ω, a universal tree can be constructed for all walks of a given digraph G. Considering only the trees $t(\kappa)$ obtained from walks with no repeated sections produces a finite number of structurally distinct trees from all walks on G. These trees can be ordered partially by inclusion and the resulting poset always admits an unique maximum. This maximum tree is of paramount importance to G: it is one of the few invariants of its hike monoid [3, 4], and dictates the shape of all branched continued fractions counting walks on G [5].

3. The co-preLie co-algebra of walks

3.1. Co-product

With the notion of admissible cut, we may now formally define the co-product associated to Lawler's process, by mapping a walk to a sum over all its admissible cuts tensored with their remainders:

Definition 7 (Co-product). Let G be a digraph. The co-product associated to Lawler's process is the linear map Δ_{CP} defined by,

$$\Delta_{\text{CP}}:\begin{cases}\mathcal{W}(G) & \to & \mathcal{W}(G)\otimes\mathcal{W}(G)\\ \omega & \mapsto & \Delta_{\text{CP}}(\omega)=\displaystyle\sum_{\omega^c\in\text{AdC}(\omega)}\omega_c\otimes\omega^c.\end{cases}$$

An essential property of this co-product is that a walk is primitive for it if and only if it is a simple path or a simple cycle.

Proposition 5. *Let G be a digraph and $\omega\in\mathcal{W}(G)$. Then,*

$$\Delta_{\text{CP}}(\omega)=0 \iff \omega \textit{ is a simple path or a simple cycle.}$$

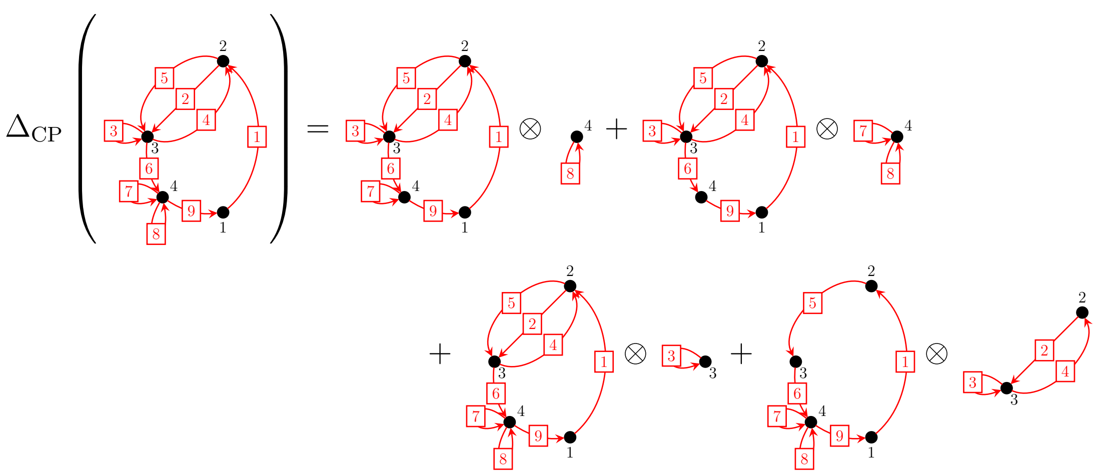
Proof. Let $\omega=w_0\dots w_\ell\in\mathcal{W}(G)$. If ω is a simple path, then it has no cycles and $\text{AdC}(\omega)=\emptyset$ so $\Delta_{\text{CP}}(\omega)=0$. If ω is a simple cycle, then $\text{LES}(\omega)=\{\omega\}$, $\text{AdC}(\omega)=\emptyset$ since the walk is not an admissible cut of itself, therefore $\Delta_{\text{CP}}(\omega)=0$.

Now suppose that ω be neither a simple path nor a simple cycle. Then ω has at least one simple cycle and we may consider the last such cycle $\omega^{k,k'}\in\text{LES}(\omega)$ erased from ω by Lawler's process. This simple cycle cannot be ω itself since ω is not a simple cycle. Furthermore after step k' no vertex of $\mathsf{Lew}_{k'}(\omega)$ is visited again in $w_{k'+1}\cdots w_\ell$ as otherwise $\omega^{k,k'}$ would not be the last erased simple cycle. This simply indicates that the last erased simple cycles is not within a wider erased section by virtue of being the last to be removed. Thus $\omega^{\min}_{k,k'}$ does not exist, $\omega^{k,k'}\in\text{AdC}(\omega)$, and $\Delta(\omega)\neq 0$. $\square$

Example 10. Consider the walk $\omega=1233234441$, then

$$\Delta_{\text{CP}}(1233234441)=123323441\otimes 44+12332341\otimes 444+123234441\otimes 33+1234441\otimes 2332,$$

or, graphically,



3.2. The co-preLie property

Having established the definition of the co-product associated to the Lawler process and identified its primitive walks, we now turn to the co-algebraic structure it gives to the walk vector space $\mathcal{W}(G)$. Recall that:

Definition 8. A co-preLie co-algebra is a couple $(\mathcal{V},\Delta)$ where $\mathcal{V}$ is a vector space and $\Delta:\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}$ is a linear map such that for any $v\in\mathcal{V}$ the following relation is satisfied [12, 6, 9]

$$(\Delta\otimes\text{Id}-\text{Id}\otimes\Delta)\circ\Delta(v)=(\text{Id}\otimes\tau)\circ(\Delta\otimes\text{Id}-\text{Id}\otimes\Delta)\circ\Delta(v)$$

where Id is the identity map and τ is the twisting linear map, $\tau:\mathcal{V}\otimes\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}$, $\tau(u\otimes v)=v\otimes u$.

Theorem 6. *The vector space $\mathcal{W}(G)$, equipped with the coproduct $\Delta_{\rm CP}$, is a co-preLie (but not co-unital) co-algebra.*

We present two proofs of this result. The first, given immediately below, is a direct approach based on the properties of admissible cuts. The second proof, presented in §5, obtains the theorem as a corollary of the Hopf structure on the tensor algebra generated by $\mathcal{W}(G)$ via a brace coalgebra construction.

Proof. Let $\omega = w_0 \dots w_\ell \in \mathcal{W}(G)$. We begin with evaluating $(\Delta_{\rm CP}\otimes {\rm Id})\circ\Delta_{\rm CP}(\omega)$ explicitly. To that end consider an admissible cut $\omega^{k,k'} \in {\rm AdC}(\omega)$, assuming that $\omega_{k,k'}$ is not self-avoiding nor a simple cycle as this leads to a 0 result. Then, $(\Delta_{\rm CP}\otimes {\rm Id})(\omega_{k,k'}\otimes\omega^{k,k'})$ yields a sum over cuts that fall into four distinct cases, depending on the second cut's coordinates l, l' relatively to k, k':

1) $l<l'<k<k'$, i.e. $\omega = w_0\cdots w_l\cdots w_{l'}\cdots w_k\cdots w_{k'}\cdots w_\ell$, this gets cut as $\omega_{k,k';l,l'}\otimes\omega^{l,l'}\otimes\omega^{k,k'}$,
2) $k<k'<l<l'$, i.e $\omega = w_0\cdots w_k\cdots w_{k'}\cdots w_l\cdots w_{l'}\cdots w_\ell$, this gets cut as $\omega_{k,k';l,l'}\otimes\omega^{l,l'}\otimes\omega^{k,k'}$,
3) $l<k<k'<l'$, i.e. $\omega = w_0\cdots w_l\cdots w_k\cdots w_{k'}\cdots w_{l'}\cdots w_\ell$, this gets cut as $\omega_{l,l'}\otimes\omega^{l,l'}_{k,k'}\otimes\omega^{k,k'}$,
4) $l<l'=k<k'$, i.e. $\omega = w_0\cdots w_l\cdots w_{l'=k}\cdots w_{k'}\cdots w_\ell$, this gets cut as $\omega_{l,k'}\otimes\omega^{lk}_{k,k'}\otimes\omega^{k,k'}$. Remark that $\omega^{l,l'}$ is not an admissible cut of ω because $w_{l'}=w_{k'}$ occurs after step l' in $\omega^{\rm min}_{l,l'}$.

By Case 1 of Proposition 2, if an admissible cut falls into situation 1) above, another one will be admissible as per situation 2). Thus,

$$(\Delta_{\rm CP}\otimes {\rm Id})\circ\Delta_{\rm CP}(\omega) = \sum_{\substack{a\in{\rm AdC}(\omega)\\ \omega_{k,k'}\notin\Pi(G)\cup\Gamma(G)}}\ \sum_{b\in{\rm AdC}(\omega_{k,k'})}\begin{cases}\omega_{k,k';l,l'}\otimes\omega^{l,l'}\otimes\omega^{k,k'} & l<l'<k<k'\\ \omega_{k,k';l,l'}\otimes\omega^{l,l'}\otimes\omega^{k,k'} & k<k'<l<l',\\ \omega_{l,l'}\otimes\omega^{k,k'}_{l,l'}\otimes\omega^{k,k'} & l<k<k'<l',\\ \omega_{l,k'}\otimes\omega^{l,k}_{k,k'}\otimes\omega^{k,k'} & l<l'=k<k',\end{cases}$$

where we used $a:=\omega^{k,k'}$ and $b:=\omega^{l,l'}$ to alleviate the notation and recall that $\Pi(G)\cup\Gamma(G)$ designates the set of simple paths and simple cycles on G.

Now we turn to $({\rm Id}\otimes\Delta_{\rm CP})\circ\Delta_{\rm CP}(\omega)$. Let again $\omega^{k,k'}\in{\rm AdC}(\omega)$ be an admissible cut of ω which we assume not to be a simple cycle as this would lead to a 0 result. Then, $({\rm Id}\otimes\Delta_{\rm CP})\circ(\omega_{k,k'}\otimes\omega^{k,k'})$ yields a sum over cuts that fall into two distinct cases, depending on the second cut's coordinates l, l' inside of $\omega_{k,k'}$:

1) $k<l<l'<k'$, that is $\omega = w_0\cdots w_k\cdots w_l\cdots w_{l'}\cdots w_{k'}\cdots w_\ell$, which gets cut as $\omega_{k,k'}\otimes\omega^{k,k'}_{l,l'}\otimes\omega^{l,l'}$,
2) $k<l<l'=k'$, that is $\omega = w_1\cdots w_k\cdots w_l\cdots w_{l'=k'}\cdots w_m$, which gets cut as $\omega_{k,k'}\otimes\omega^{k,k'}_{l,k'}\otimes\omega^{l,k'}$.

Here we do not need to consider the case $k=l<l'\le k'$. Indeed, either $k=l<l'=k'$ then $\omega^{l,l'}=\omega^{k,k'}$ meaning we cut $\omega^{k,k'}$ out of itself, which is not admissible; or $k=l<l'<k'$ but then, $\omega^{l,l'}$ is not admissible because $w_{l'}=w_{k'}$ is visited again after step l'. Rather, in that situation it is $\omega^{l',k'}$ that is admissible and falls into case 2) above. Thus,

$$({\rm Id}\otimes\Delta_{\rm CP})\circ\Delta_{\rm CP}(\omega) = \sum_{\substack{a\in{\rm AdC}(\omega)\\ a\notin\Gamma(G)}}\ \sum_{b\in{\rm AdC}(\omega^{k,k'})}\begin{cases}\omega_{k,k'}\otimes\omega^{k,k'}_{l,l'}\otimes\omega^{l,l'} & k<l<l'<k',\\ \omega_{k,k'}\otimes\omega^{k,k'}_{l,k'}\otimes\omega^{l,k'} & k<l<l'=k',\end{cases}$$

where $a := \omega^{k,k'}$ and $b := \omega^{l,l'}$. By Case 2 of Proposition 2, gathering everything, we obtain

$$(\Delta_{\mathrm{CP}} \otimes \mathrm{Id} - \mathrm{Id} \otimes \Delta_{\mathrm{CP}}) \circ \Delta_{\mathrm{CP}}(\omega) = \sum_{\substack{a\in \mathrm{AdC}(\omega) \\ \omega_{k,k'}\notin \Pi(G)\cup\Gamma(G)}} \sum_{\substack{b\in \mathrm{AdC}(\omega_{k,k'}) \\ l<l'<k<k' \\ k<k'<l<l'}} \omega_{k,k';l,l'} \otimes \omega^{l,l'} \otimes \omega^{k,k'}.$$

Remark how k, k' and l, l' now play completely symmetric roles in the above so that the co-prelie relation holds for all walks $\omega \in \mathcal{W}(G)$,

$$(\Delta_{\mathrm{CP}} \otimes \mathrm{Id} - \mathrm{Id} \otimes \Delta_{\mathrm{CP}}) \circ \Delta_{\mathrm{CP}}(\omega) = (\mathrm{Id} \otimes \tau) \circ (\Delta_{\mathrm{CP}} \otimes \mathrm{Id} - \mathrm{Id} \otimes \Delta_{\mathrm{CP}}) \circ \Delta_{\mathrm{CP}}(\omega).$$

This indicates, perhaps suprisingly, that Lawler's intuitive chronological removal of the simple cycles from walks naturally endows their vector space with a sophisticated co-preLie structure. □

Example 11. Consider again the walk $\omega = 1233234441$ of Example 10. Then,

$$\begin{aligned}\big(\Delta_{\mathrm{CP}} \otimes \mathrm{Id}\big) \circ \Delta_{\mathrm{CP}}(1233234441) &= 12332341 \otimes 44 \otimes 44 + 12323441 \otimes 33 \otimes 44 + 123441 \otimes 2332 \otimes 44 \\ &\quad + 1232341 \otimes 33 \otimes 444 + 12341 \otimes 2332 \otimes 444 \\ &\quad + 12323441 \otimes 44 \otimes 33 + 1232341 \otimes 444 \otimes 33 + 1234441 \otimes 232 \otimes 33 \\ &\quad + 123441 \otimes 44 \otimes 2332 + 12341 \otimes 444 \otimes 2332, \\ \big(\mathrm{Id} \otimes \Delta_{\mathrm{CP}}\big) \circ \Delta_{\mathrm{CP}}(1233234441) &= 12332341 \otimes 44 \otimes 44 + 1234441 \otimes 232 \otimes 33.\end{aligned}$$

From this, reordering the terms for convenience, we obtain

$$\begin{aligned}\big(\Delta_{\mathrm{CP}} \otimes \mathrm{Id} - \mathrm{Id} \otimes \Delta_{\mathrm{CP}}\big) \circ \Delta_{\mathrm{CP}}(1233234441) &= 12323441 \otimes 33 \otimes 44 + 12323441 \otimes 44 \otimes 33 \\ &\quad + 123441 \otimes 2332 \otimes 44 + 123441 \otimes 44 \otimes 2332 \\ &\quad + 1232341 \otimes 33 \otimes 444 + +1232341 \otimes 444 \otimes 33 \\ &\quad + 12341 \otimes 2332 \otimes 444 + 12341 \otimes 444 \otimes 2332,\end{aligned}$$

which is invariant under the action of $\mathrm{Id} \otimes \tau$ as dictated by Theorem 6.

4. Hopf structures on the tensor and symmetric algebras of walks

As in the case of trees due to Connes and Kreimer in [1] or the case of decorated trees due to Foissy [2], for any walk, we can extend the notion of admissible cut to allow for multiple simultaneous cuts, which we call *extended* admissible cuts. Thanks to this construction, a dual of Oudon and Guin's own [10], we get a co-product compatible with the tensor and symmetric algebra structures generated by walks on G.

4.1. Extended admissible cuts

We begin by defining the notion of extended admissible cuts of a walk, then show that they turn the tensor and symmetric algebras of walks into Hopf algebras. Finally, we present three special families of walks, ladders, corollas and cacti, and explain how we can make them into Hopf algebras.

Definition 9. Let G be a finite connected non-empty graph and $\mathbb{K}$ a field of characteristic 0.

1. We define $\mathcal{T}\langle\mathcal{W}(G)\rangle$ as the tensor algebra generated by $\mathcal{W}(G)$. To alleviate the notation, for walks $\omega_1, \ldots, \omega_p \in \mathcal{W}(G)$, the tensor $\omega_1 \otimes \cdots \otimes \omega_p$ will be denoted by $\omega_1 \,|\, \ldots \,|\, \omega_p$. Such elements of $\mathcal{T}\langle\mathcal{W}(G)\rangle$ are called forests.

2. Let $\omega = w_0 \dots w_\ell$ be a walk in G. In keeping with common terminology for Hopf algebras we call *degree* $\deg(\omega)$ the length of the walk ω, i.e. $\deg(\omega) = \ell$

We recall that the tensor algebra $\mathcal{T}\langle\mathcal{W}(G)\rangle$ is equipped with the concatenation product $\bullet$,

$$\bullet : \begin{cases} \mathcal{T}\langle\mathcal{W}(G)\rangle \otimes \mathcal{T}\langle\mathcal{W}(G)\rangle & \longrightarrow & \mathcal{T}\langle\mathcal{W}(G)\rangle \\ \omega_1 \,|\, \dots \,|\, \omega_m \otimes \omega'_1 \,|\, \dots \,|\, \omega'_n & \longmapsto & \omega_1 \,|\, \dots \,|\, \omega_m \,|\, \omega'_1 \,|\, \dots \,|\, \omega'_n, \end{cases}$$

and the degree $\deg(\omega_1|\dots|\omega_m) = \sum_{i=1}^{m} \deg(\omega_i)$ is the sum of the involved walks' degrees. By construction, $(\mathcal{T}\langle\mathcal{W}(G)\rangle, \bullet)$ is an unital associative algebra with unit the empty forest (), identified with $\mathbf{1} \in \mathbb{K}$, written in bold font so as to distinguish it from a vertex label '1'.

Remark 6. Seeing the walks as words on vertices and setting their degrees to be the number of vertices in these words leads to inconsistencies with the coproduct Δ_{CP}, e.g. $\Delta_{\text{CP}}(w_1w_1w_1) = w_1w_1 \otimes w_1w_1$ but $\deg(w_1w_1) = 2$ so that $\deg(w_1w_1) + \deg(w_1w_1) = 4$ yet $\deg(w_1w_1w_1) = 3$. Ultimately, this indicates that walks really ought to be seen as words on edges rather than vertices, something consistent with the Cartier-Foata monoids of hikes [4].

Definition 10 (Extended admissible cut). Let G be a digraph and $\omega \in \mathcal{W}(G)$. An *extended admissible cut* of ω is the tensor product of $n \in \mathbb{N}\backslash\{0\}$ consecutive admissible cuts $\omega^{k_i,k'_i} \in \text{AdC}(\omega)$ which are *non-overlapping* in ω, that is $k_1 < k'_1 < k_2 < k'_2 < \dots < k_n < k'_n$. We write,

$$\omega^{k_1,k'_1;\dots;k_n,k'_n} := \omega^{k_1,k'_1} \,|\, \dots \,|\, \omega^{k_n,k'_n} \in \mathcal{T}\langle\mathcal{W}(G)\rangle.$$

The set of extended admissible cuts of ω is denoted $E\text{AdC}(\omega)$. Observe that $\text{AdC}(\omega) \subset E\text{AdC}(\omega)$.

To alleviate the notation whenever possible we designate an extended admissible cut by a single letter, e.g. $\omega^c \in E\text{AdC}(\omega)$ and might then simply write that c is an extended admissible cut of ω.

Example 12. Consider the walk $\omega = 123324441$, then

$$E\text{AdC}(\omega) = \{33, 44, 444, 2332, 33 \,|\, 44, 33 \,|\, 444, 2332 \,|\, 44, 2332 \,|\, 444\}.$$

As stated Definition 10 the admissible cuts constituting an extended admissible cut ω^c of a walk $\omega = w_0 \cdots w_\ell$ are non-overlapping, i.e. $\omega^c := \omega^{k_1,k'_1;\dots;k_n,k'_n} \in E\text{AdC}(\omega)$ satisfies
$0 \leq k_1 < k'_1 < k_2 < k'_2 < \dots < k_n < k'_n \leq \ell$. We can therefore meaningfully denote

$$\omega_c := \omega_{k_1,k'_1;\dots;k_n,k'_n} = w_0 \dots w_{k_1} w_{k'_1+1} \dots w_{k_2} w_{k'_2+1} \dots w_{k_n} w_{k'_n+1} \dots w_\ell,$$

for what remains of ω after erasure of all ω^{k_i,k'_i}. Since admissible cuts are closed subwalks of a walk, ω_c is still a walk. Together with the non-overlapping condition this implies that, for any $1 \leq i \leq n$,

$$\omega^{k_1,k'_1;\dots;k_n,k'_n} \in E\text{AdC}(\omega) \Rightarrow \omega^{k_i,k'_i} \in \text{AdC}(\omega_{k_1,k'_1;\dots;k_{i-1},k'_{i-1};k_{i+1},k'_{i+1};\dots;k_n,k'_n}). \quad (3)$$

Extended admissible cuts are 'well behaved' in the sense that such cuts and their remainders satisfy an analog of Proposition 2 for admissible cuts:

Proposition 7. *Let G a digraph, $\omega \in \mathcal{W}(G)$ a walk, $\omega^c \in E\text{AdC}(\omega)$. Then,*

$$\omega^{c'} \in \text{AdC}(\omega_c) \Rightarrow \omega^{c'} \in \text{AdC}(\omega),$$

which also implies

$$\omega^{c'} \in E\text{AdC}(\omega_c) \Rightarrow \omega^{c'} \in E\text{AdC}(\omega).$$

Furthermore,

$$\omega^{c'} \in E\text{AdC}(\omega^c) \Rightarrow \omega^{c'} \in E\text{AdC}(\omega).$$

Proof. Let $\omega^c := \omega^{k_1,k_1';\dots;k_n,k_n'}$ be an extended admissible cut of ω. By virtue of Proposition 2, an admissible cut of the remainder of an admissible cut of a walk is an admissible cut of that walk,

$$\omega^{k,k'} \in \mathrm{AdC}(\omega),\ \omega^{l,l'} \in \mathrm{AdC}(\omega_{k,k'}) \Rightarrow \omega^{l,l'} \in \mathrm{AdC}(\omega).$$

In addition $\omega^{k_1,k_1'}_{k_2,k_2';\dots;k_n,k_n'} \in \mathrm{AdC}(\omega_{k_2,k_2';\dots;k_n,k_n'})$ by virtue of the fact that extended admissible cuts only comprise non-overlapping admissible cuts. Therefore we get

$$\omega^{l,l'} \in \mathrm{AdC}(\omega_{k_1,k_1';\dots;k_n,k_n'}) \Rightarrow \omega^{l,l'} \in \mathrm{AdC}(\omega_{k_2,k_2';\dots;k_n,k_n'}).$$

Iterating this observation leads to the first claim for $\omega^{c'} := \omega^{l,l'}$. Note that since all cuts are non-overlapping we could have chosen to remove the k_i, k_i' cuts in any order in the iteration. The result for extended admissible cuts is now immediate since such cuts comprise only non-overlapping admissible cuts to each of which we apply the result just proven.

For the second claim, observe that by Case 2 of Proposition 2, $\omega^c \in \mathrm{AdC}(\omega)$ and $\omega^{c'} \in \mathrm{AdC}(\omega^c)$ implies $\omega^{c'} \in \mathrm{AdC}(\omega)$. The result for extended admissible cuts follows once more from the observation that such cuts comprise only non-overlapping admissible cuts, each of which behaves as dictated by Case 2 of Proposition 2. □

4.2. Hopf algebra on walks

We may now define a co-product on $\mathcal{T}\langle\mathcal{W}(G)\rangle$ by relying on extended admissible cuts and their remainders:

Definition 11 (Extended co-product). Let G be a digraph. Consider the morphism of algebras Δ_{H} defined by:

$$\Delta_{\mathrm{H}} : \begin{cases} \mathcal{T}\langle\mathcal{W}(G)\rangle & \longrightarrow \quad \mathcal{T}\langle\mathcal{W}(G)\rangle \otimes \mathcal{T}\langle\mathcal{W}(G)\rangle \\ \qquad \omega & \longmapsto \quad \Delta_{\mathrm{H}}(\omega) = \mathbf{1}\otimes\omega + \omega\otimes\mathbf{1} + \displaystyle\sum_{c\in E\mathrm{AdC}(\omega)} \omega_c \otimes \omega^c, \end{cases}$$

where ω is a walk and the sum runs over all extended admissible cuts ω^c of ω.

Theorem 8. *Let G a finite digraph and consider the triple $\mathcal{H}_{\mathcal{T}} := (\mathcal{T}\langle\mathcal{W}(G)\rangle, \bullet, \Delta_{\mathrm{H}})$. Equipped with the map* deg*, it defines a graded connected Hopf algebra.*

Proof. Observe first that deg is a graduation by direct calculation. Second, to prove the theorem we must establish that Δ_{H} is coassociative. Since Δ_{H} is a morphism of algebras it is sufficient to show that for any walk $\omega \in \mathcal{W}(G)$,

$$(\Delta_{\mathrm{H}} \otimes \mathrm{Id}) \circ \Delta_{\mathrm{H}}(\omega) = (\mathrm{Id} \otimes \Delta_{\mathrm{H}}) \circ \Delta_{\mathrm{H}}(\omega).$$

Let ω be a walk in G. Then

$$\begin{aligned}(\Delta_{\mathrm{H}} \otimes \mathrm{Id}) \circ \Delta_{\mathrm{H}}(\omega) &= \omega\otimes\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\omega\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{1}\otimes\omega + \sum_{c\in E\mathrm{AdC}(\omega)} \omega_c\otimes\omega^c\otimes\mathbf{1} \\ &\quad + \sum_{c\in E\mathrm{AdC}(\omega)} \omega_c\otimes\mathbf{1}\otimes\omega^c + \sum_{c\in E\mathrm{AdC}(\omega)} \mathbf{1}\otimes\omega_c\otimes\omega^c + \sum_{c\in E\mathrm{AdC}(\omega)}\sum_{c'\in E\mathrm{AdC}(\omega_c)} (\omega_c)_{c'}\otimes(\omega_c)^{c'}\otimes\omega^c,\end{aligned}$$

Similarly,

$$\begin{aligned}(\mathrm{Id} \otimes \Delta_{\mathrm{H}}) \circ \Delta_{\mathrm{H}}(\omega) &= \omega\otimes\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\omega\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{1}\otimes\omega + \sum_{c\in E\mathrm{AdC}(\omega)} \mathbf{1}\otimes\omega_c\otimes\omega^c \\ &\quad + \sum_{c\in E\mathrm{AdC}(\omega)} \omega_c\otimes\mathbf{1}\otimes\omega^c + \sum_{c\in E\mathrm{AdC}(\omega)} \omega_c\otimes\omega^c\otimes\mathbf{1} + \sum_{c\in E\mathrm{AdC}(\omega)}\sum_{c'\in E\mathrm{AdC}(\omega^c)} \omega_c\otimes(\omega^c)_{c'}\otimes(\omega^c)^{c'},\end{aligned}$$

So the theorem follows if we prove that

$$\sum_{c\in E\mathrm{AdC}(\omega)}\ \sum_{c'\in E\mathrm{AdC}(\omega_c)} (\omega_c)_{c'} \otimes (\omega_c)^{c'} \otimes \omega^c = \sum_{c\in E\mathrm{AdC}(\omega)}\ \sum_{c'\in E\mathrm{AdC}(\omega^c)} \omega_c \otimes (\omega^c)_{c'} \otimes (\omega^c)^{c'}. \tag{4}$$

Consider first terms from the left-hand side (LHS) of the above, i.e. of the form

$$(\omega_c)_{c'} \otimes (\omega_c)^{c'} \otimes \omega^c \tag{5}$$

with $c \in E\mathrm{AdC}(\omega)$ and $c' \in E\mathrm{AdC}(\omega_c)$.

$$\omega_k \otimes (\omega^k)_c \otimes \omega^c$$

with $\omega^k \in E\mathrm{AdC}(\omega)$ and $\omega^c \in E\mathrm{AdC}(\omega^k)$. This implies that the LHS of Eq. (4) is comprised in its right-hand side (RHS).

Second, consider terms from the RHS of Eq. (4),

$$\omega_c \otimes (\omega^c)_{c'} \otimes (\omega^c)^{c'}, \tag{6}$$

with $c \in E\mathrm{AdC}(\omega)$ and $c' \in E\mathrm{AdC}(\omega^c)$. Since $c \in E\mathrm{AdC}(\omega)$ and $c' \in E\mathrm{AdC}(\omega^c)$ then $c' \in EAdC(\omega)$ by the second result of Proposition 7. Since $c' \in E\mathrm{AdC}(\omega^c)$, c' is entirely included within cut c and we can define $l := c\backslash c'$, $c = l \cup c'$ to be the extended admissible cut $\omega^l \in E\mathrm{AdC}(\omega_{c'})$ which cuts out c from the remainder $\omega_{c'}$. By construction $(\omega_{c'})^l = (\omega^c)_{c'}$ and $(\omega_{c'})_l = \omega_{c,c'}$. Consequently, any term of the form given by Eq. (6) is also of the form

$$(\omega_{c'})_l \otimes (\omega_{c'})^l \otimes \omega^{c'}$$

with $\omega^{c'} \in E\mathrm{AdC}(\omega)$ and $\omega^l \in E\mathrm{AdC}(\omega_{c'})$. This implies that the RHS of Eq. (4) is comprised in its LHS.

The equality of Eq. (4) is proven and Δ_{H} is coassociative. Thus we have a graded connected bialgebra hence a Hopf algebra. □

Remark 7. In the proof above, we showed that the LHS of Eq. (4) is comprised in its RHS. Observe that this is not true for admissible cuts, indeed we used that $k := c \cup c'$ is the union of two non-overlapping cuts and so while $k \in E\mathrm{AdC}(\omega)$, we have $k \notin \mathrm{AdC}(\omega)$. This explains why Δ_{CP} fails to be coassociative.

At the opposite, it would still be true that the RHS of Eq. (4) is comprised in its LHS had we allowed only for admissible cuts. This is because if $c' \in \mathrm{AdC}(\omega)$ and $c \in \mathrm{AdC}(\omega_{c'})$ then $l := c\backslash c'$ is an admissible cut of $\mathrm{AdC}(\omega_{c'})$ by Proposition 2. This indicates that all terms generated by $\big(\mathrm{Id} \otimes \Delta_{\mathrm{CP}}\big) \circ \Delta_{\mathrm{CP}}$ can be found in those generated by $\big(\Delta_{\mathrm{CP}} \otimes \mathrm{Id}\big) \circ \Delta_{\mathrm{CP}}$, see e.g. Example 11

Remark 8. Note that we can always see any finite walk on an infinite digraph as a walk on a finite digraph. As a corollary, properties of the product, co-product, unit, co-unit and existence of the antipode extend to the case of walks on infinite digraphs where $\mathcal{H}_{\mathcal{T}}$ is still a Hopf algebra.

Corollary 9. *Let G a digraph. The triple $\mathcal{H}_{\mathcal{T}} := (\mathcal{T}\langle\mathcal{W}(G)\rangle, \bullet, \Delta_{\mathrm{H}})$ is a Hopf algebra.*

Example 13. Let $\omega = 1233234441$ be the walk of Examples 10, 11 and 12 and consider the reduced co-product $\overline{\Delta}_{\mathrm{H}}(\omega) := \Delta_{\mathrm{H}}(\omega) - \mathbf{1} \otimes \omega - \omega \otimes \mathbf{1}$. Then

$$\begin{aligned}\overline{\Delta}_{\mathrm{H}}(\omega) &= 123323441 \otimes 44 + 12332341 \otimes 444 + 123234441 \otimes 33 + 1234441 \otimes 2332\\ &\quad + 12323441 \otimes 33 \,|\, 44 + 1232341 \otimes 33 \,|\, 444 + 123441 \otimes 2332 \,|\, 44 + 12341 \otimes 2332 \,|\, 444.\end{aligned}$$

Thus we have,

$$(\overline{\Delta}_{\mathrm{H}} \otimes \mathrm{Id}) \circ \overline{\Delta}_{\mathrm{H}}(\omega) =$$

$$
\begin{aligned}
&12332341 \otimes 44 \otimes 44 + 12323441 \otimes 33 \otimes 44 + 123441 \otimes 2332 \otimes 44 + 1232341 \otimes 33 \,|\, 44 \otimes 44 \\
&\qquad + 12341 \otimes 2332 \,|\, 44 \otimes 44 \\
&+ 1232341 \otimes 33 \otimes 444 + 12341 \otimes 2332 \otimes 444 \\
&+ 12323441 \otimes 44 \otimes 33 + 1232341 \otimes 444 \otimes 33 + 1234441 \otimes 232 \otimes 33 + 123441 \otimes 232 \,|\, 44 \otimes 33 \\
&\qquad + 12341 \otimes 232 \,|\, 444 \otimes 33 \\
&+ 12341 \otimes 444 \otimes 2332 + 123441 \otimes 44 \otimes 2332 \\
&+ 12341 \otimes 232 \,|\, 44 \otimes 33 \,|\, 44 + 123441 \otimes 232 \otimes 33 \,|\, 44 + 1232341 \otimes 44 \otimes 33 \,|\, 44 \\
&+ 12341 \otimes 232 \otimes 33 \,|\, 444 \\
&+ 12341 \otimes 44 \otimes 2332 \,|\, 44.
\end{aligned}
$$

The presentation has been organized for the sake of readability: each line above represents terms stemming from the same term found in $\overline{\Delta}_{\mathrm{H}}(\omega)$, while an additional indentation denotes a continuing line. Similarly,

$$
\begin{aligned}
(\mathrm{Id} \otimes \overline{\Delta}_{\mathrm{H}}) \circ \overline{\Delta}_{\mathrm{H}}(\omega) &= 12332341 \otimes 44 \otimes 44 \\
&+ 1234441 \otimes 232 \otimes 33 \\
&+ 12323441 \otimes 44 \otimes 33 + 12323441 \otimes 33 \otimes 44 \\
&+ 1232341 \otimes 33 \otimes 444 + 1232341 \otimes 444 \otimes 33 + 1232341 \otimes 33 \,|\, 44 \otimes 44 + 1232341 \otimes 44 \otimes 33 \,|\, 44 \\
&+ 123441 \otimes 2332 \otimes 44 + 123441 \otimes 232 \,|\, 44 \otimes 33 + 123441 \otimes 232 \otimes 33 \,|\, 44 + 123441 \otimes 44 \otimes 2332 \\
&+ 12341 \otimes 232 \,|\, 444 \otimes 33 + 12341 \otimes 2332 \,|\, 44 \otimes 44 + 12341 \otimes 232 \,|\, 44 \otimes 33 \,|\, 44 + 12341 \otimes 2332 \otimes 444 \\
&\qquad + 12341 \otimes 444 \otimes 2332 + 12341 \otimes 232 \otimes 33 \,|\, 444 + 12341 \otimes 2332 \,|\, 44.
\end{aligned}
$$

A *close* examination of both results reveals their equality as predicted by Theorem 8.

Let G be a digraph. Let G be a finite connected non-empty graph. We denote by $\mathcal{I}$ the vector space spanned by the elements $\omega_1|\dots|\omega_n - \omega_{\sigma(1)}|\dots|\omega_{\sigma(n)}$ where $\omega_1|\dots|\omega_n \in \mathcal{T}\langle\mathcal{W}(G)\rangle$ and σ is a permutation. We define

$$
\mathcal{S}\langle\mathcal{W}(G)\rangle := \frac{\mathcal{T}\langle\mathcal{W}(G)\rangle}{\mathcal{I}}.
$$

In the vector space $\mathcal{S}\langle\mathcal{W}(G)\rangle$ the concatenation product $\bullet$ becomes the disjoint-union product $\square$. By construction, $\mathcal{S}\langle\mathcal{W}(G)\rangle$ is the Abelianization of the Hopf algebra $\mathcal{T}\langle\mathcal{W}(G)\rangle$ and therefore :

Corollary 10. *Let G be a digraph. Then $\mathcal{H}_{\mathcal{S}} := (\mathcal{S}\langle\mathcal{W}(G)\rangle, \square, \Delta_{\mathrm{H}})$ is a Hopf algebra.*

4.3. Antipode

In this section we construct the antipodes explicitly, relying on the total time-order introduced in Definition 5.

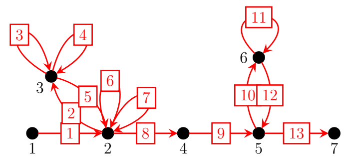
Definition 12. Let ω be a walk, $\mathrm{AdC}(\omega)$ be its set of admissible cuts which we assume to be not empty. Let $1 \le n \le |\mathrm{AdC}(\omega)|$ be a positive integer, $c_i \in \mathrm{AdC}(\omega)$ a collection of n totally ordered, distinct, non-overlapping admissible cuts of ω with $c_1 \leqslant_{🕒} \cdots \leqslant_{🕒} c_n$. Let $e := c_1 \,|\, \dots \,|\, c_n$, we may also conveniently use the notation $|e| := n$. We associate to e a tensor T_e and a disjoint union S_e as follows,

$$
T_e := \omega_{c_1,\dots,c_n} \,|\, (\omega_{c_1,\dots,c_{n-1}})^{c_n} \,|\, \cdots \,|\, (\omega_{c_1,\dots,c_{i-1}})^{c_i} \,|\, \cdots \,|\, (\omega_{c_1})^{c_2} \,|\, \omega^{c_1},
$$

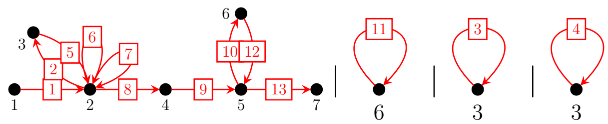
and

$$
S_e := \omega_{c_1,\dots,c_n} \,\square\, (\omega_{c_1,\dots,c_{n-1}})^{c_n} \,\square\, \cdots \,\square\, (\omega_{c_1,\dots,c_{i-1}})^{c_i} \,\square\, \cdots \,\square\, (\omega_{c_1})^{c_2} \,\square\, \omega^{c_1}.
$$

Example 14. Consider again the walk of Example 8,

$\omega = 12333222456657 =$ 

and three of its admissible cuts $c_1 = \omega^{3,4}$, $c_2 = \omega^{2,4}$, and $c_3 = \omega^{10,11}$. Since $\omega^{3,4} \leqslant_{\circlearrowright} \omega^{2,4} \leqslant_{\circlearrowright} \omega^{10,11}$, for $e := c_1 \,|\, c_2 \,|\, c_3$, $|e| = 3$,

$T_e =$ 

Theorem 11. *Let G be a digraph and $\omega \in \mathcal{W}(G)$ a walk. Then, in $\mathcal{T}\langle\mathcal{W}(G)\rangle$, the antipode $S(\omega)$ calculated on ω is,*

$$S(\omega) = -\omega - \sum_{e \in E\mathrm{AdC}(\omega)} (-1)^{|e|} T_e = -\omega - \sum_{n=1}^{|\mathrm{AdC}(\omega)|} \sum_{\substack{c_1 <_{\circlearrowright} \cdots <_{\circlearrowright} c_n \\ c_i \in \mathrm{AdC}(\omega)}} (-1)^n \, T_{c_1|\ldots|c_n}$$

where $|\mathrm{AdC}(\omega)|$ designates the cardinality of $\mathrm{AdC}(\omega)$.

Corollary 12. *Let G be a digraph and $\omega \in \mathcal{W}(G)$ a walk. Then, in $\mathcal{S}\langle\mathcal{W}(G)\rangle$, the antipode $S(\omega)$ calculated on ω is,*

$$S(\omega) = -\omega - \sum_{e \in E\mathrm{AdC}(\omega)} (-1)^{|e|} S_e = -\omega - \sum_{n=1}^{|\mathrm{AdC}(\omega)|} \sum_{\substack{c_1 <_{\circlearrowright} \cdots <_{\circlearrowright} c_n \\ c_i \in \mathrm{AdC}(\omega)}} (-1)^n \, S_{c_1|\ldots|c_n} \, ,$$

where $|\mathrm{AdC}(\omega)|$ designates the cardinality of $\mathrm{AdC}(\omega)$.

Proof of Theorem 11. We prove the theorem by induction on the cardinality of $\mathrm{AdC}(\omega)$, using the relation $\varepsilon = \bullet \circ (\mathrm{Id} \otimes S) \circ \Delta_{\mathrm{H}}$ where ε is the counity of the Hopf algebra $\mathcal{T}\langle\mathcal{W}(G)\rangle$, and the algebra antimorphism relation $S(\omega|\omega') = S(\omega')S(\omega)$ for ω, ω' walks.

Firstly, if $\mathrm{AdC}(\omega) = \emptyset$ then ω is either a simple path or a simple cycle and therefore $S(\omega) = -\omega$.

Secondly, if $\mathrm{AdC}(\omega) = \{\omega^{k,k'}\}$ then by Proposition 2, $\omega_{k,k'}$is either a simple path or a simple cycle and

$$\Delta_{\mathrm{H}}(\omega) = \omega \otimes \mathbf{1} + \mathbf{1} \otimes \omega + \omega_{k,k'} \otimes \omega^{k,k'}.$$

Consequently,

$$S(\omega) = -\omega + \omega_{k,k'} \otimes \omega^{k,k'},$$

as claimed by the theorem.

Thirdly, we assume that there exists and integer $n \in \mathbb{N}$ such that the theorem is satisfied by any walk $\omega' \in \mathcal{W}(G)$ with $|\mathrm{AdC}(\omega')| \leq n$. Consider $\omega \in \mathcal{W}(G)$ a walk with $|\mathrm{AdC}(\omega)| = n + 1$. Then,

$$S(\omega) = -\,\omega - \sum_{\substack{k_1 < k_1' < \cdots < k_n < k_n' \\ \omega^{k_i, k_i'} \in \mathrm{AdC}(\omega)}} \omega_{k_1,k_1';\ldots;k_n,k_n'} \bullet S(\omega^{k_1,k_1'} \bullet \ldots \bullet \omega^{k_n,k_n'})$$

$$= -\,\omega - \sum_{\substack{k_1 < k_1' < \dots < k_n < k_n' \\ \omega^{k_i,k_i'} \in \mathrm{AdC}(\omega)}} \omega_{k_1,k_1';\dots;k_n,k_n'} \bullet S(\omega^{k_s,k_s'}) \bullet \dots \bullet S(\omega^{k_1,k_1'}).$$

Thanks to Proposition 2,

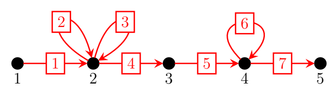
$$\bigcup_{i=1}^{n} \mathrm{AdC}(\omega^{k_i,k_i'}) \subset \mathrm{AdC}(\omega).$$

and as a consequence, $\forall i \in \{1,\dots,n\}$, $|\mathrm{AdC}(\omega^{k_i,k_i'})| \leq n$ and by induction hypothesis the theorem holds true for all $\omega^{k_i,k_i'}$. In particular, since any collection of m admissible cuts of any $\omega^{k_i,k_i'}$ is totally ordered by $\leqslant_{⌚}$ per Proposition 3,

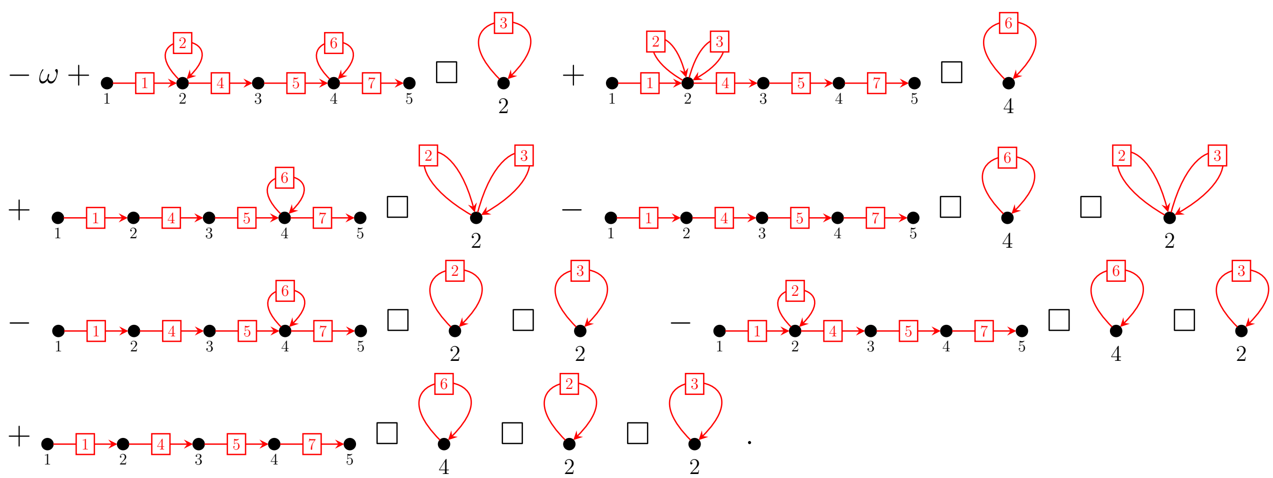
$$S(\omega) = -\omega - \sum_{n=1}^{|\mathrm{AdC}(\omega)|} \sum_{\substack{\omega^{k_1,k_1'} <_{⌚} \dots <_{⌚} \omega^{k_n,k_n'} \\ \omega^{k_i,k_i'} \in \mathrm{AdC}(\omega)}} (-1)^n T_{\omega^{k_1,k_1'} | \dots | \omega^{k_n,k_n'}}.$$

□

Example 15. Consider the walk

$\omega = 12223445 =$ 

which has three admissible cuts $\mathrm{AdC}(\omega) = \{\omega^{2,3}, \omega^{1,3}, \omega^{5,6}\}$ with $\omega^{2,3} \leqslant_{⌚} \omega^{1,3} \leqslant_{⌚} \omega^{5,6}$. Then the antipode of ω is

$S(\omega) = -\,\omega +$ 

5. Brace coalgebra and codendriform bialgebra on walks

5.1. Brace coalgebra

We show in this section that by paying attention to the number of admissible cuts appearing simultaneously in extended admissible cuts, we may endow $\mathcal{T}\langle\mathcal{W}(G)\rangle$ with a brace coalgebra structure from which the preLie co-structure on $\mathcal{W}(G)$ is recovered. We begin by recalling the necessary definitions pertaining to brace coalgebras.

Definition 13 (B_∞-algebra). Let $\mathcal{V}$ be a vector space, $\mathcal{T}\langle\mathcal{V}\rangle$ the tensor algebra generated by $\mathcal{V}$ and let π be the canonical projection from $\mathcal{T}\langle\mathcal{V}\rangle$ to $\mathcal{V}$. A B_∞-algebra is a family $(\mathcal{V}, (\langle -,-\rangle_{k,l})_{k,l\geq 0})$ where $\mathcal{V}$ is a vector space and for any $k,l \geq 0$, $\langle -,-\rangle_{k,l} : \mathcal{V}^{\otimes k} \otimes \mathcal{V}^{\otimes l} \longrightarrow \mathcal{V}$ such that:

i) $\langle -,-\rangle_{k,0} = \langle -,-\rangle_{0,k} = 0$ if $k \neq 1$ and $\langle -,-\rangle_{1,0} = \langle -,-\rangle_{0,1} = \mathrm{Id}_{\mathcal{V}}$.

ii) The unique coalgebra morphism $m : \mathcal{T}\langle\mathcal{V}\rangle \otimes \mathcal{T}\langle\mathcal{V}\rangle \longrightarrow \mathcal{T}\langle\mathcal{V}\rangle$ defined by $\pi \circ m_{\mathcal{V}^{\otimes k} \otimes \mathcal{V}^{\otimes l}} = \langle -,-\rangle_k$ is associative.

Then, equipped with the deconcatenation coproduct $\Delta_{\mathrm{dec}}(v_1 \dots v_n) := \sum_{i=0}^{n} v_1 \dots v_i \otimes v_{i+1} \dots v_n$, the triple $(\mathcal{T}\langle\mathcal{V}\rangle, m, \Delta_{\mathrm{dec}})$ is a Hopf algebra.

A *brace algebra* is a B_∞-algebra such that $\langle -,-\rangle_{k,l} = 0$ if $k \geq 2$. If $\mathcal{V}$ is a brace algebra, then for any $u = x_1 \dots x_k \in \mathcal{V}^{\otimes k}$ and $v \in \mathcal{T}\langle\mathcal{V}\rangle_+$,

$$m(u \otimes v) = \sum_{v = v_0 \dots v_{2k}} v_0 \langle x_1, v_1\rangle v_2 \dots \langle x_k, v_{2k-1}\rangle v_{2k},$$

where v_i may be empty.

Dually, a locally finite brace coalgebra is a family $(\mathcal{V}, (\delta_n)_{n\geq 1})$ where $\mathcal{V}$ is a vector space and for any n, $\delta_n : \mathcal{V} \longrightarrow \mathcal{V} \otimes \mathcal{V}^{\otimes n}$; for any $v \in \mathcal{V}$, there exists $N(v) \in \mathbb{N}$ such that if $n \geq N(v)$, then $\delta_n(v) = 0$; and the algebra morphism defined by

$$\Delta : \begin{cases} \mathcal{V} & \longrightarrow & \mathcal{T}\langle\mathcal{V}\rangle \otimes \mathcal{T}\langle\mathcal{V}\rangle \\ v & \longmapsto & \Delta(v) = 1 \otimes v + v \otimes 1 + \sum_{n\geq 1} \underbrace{\delta_n(v)}_{\in \mathcal{V} \otimes \mathcal{V}^{\otimes n} \subseteq \mathcal{T}\langle\mathcal{V}\rangle \otimes \mathcal{T}\langle\mathcal{V}\rangle} \end{cases}, \tag{7}$$

is coassociative. The co-product then extends multiplicatively to $\mathcal{T}\langle\mathcal{V}\rangle$; and $(\mathcal{T}\langle\mathcal{V}\rangle, \bullet, \Delta)$ is a bialgebra, $\bullet$ being the concatenation product.

Proposition 13. *Let $(\mathcal{V}, (\delta_n)_{n\geq 0})$ be a brace coalgebra with Δ the associated coassociative coproduct on $\mathcal{T}\langle\mathcal{V}\rangle$ as defined in Eq. (7). Then $(\mathcal{V}, \delta_1)$ is a preLie coalgebra.*

Proof. Let π be the canonical projection from $\mathcal{T}\langle\mathcal{V}\rangle$ to $\mathcal{V}$. For any $v_1, \dots, v_n \in \mathcal{V}$,

$$(\pi \otimes \pi) \circ \Delta(v_1 \dots v_n) = \begin{cases} \delta_1(v_1), & \text{if } n = 1, \\ v_1 \otimes v_2 + v_2 \otimes v_1, & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for any $v \in \mathcal{V}$,

$$\begin{aligned} (\pi \otimes \pi \otimes \pi) \circ (\Delta \otimes \mathrm{Id}) \circ \Delta(v) &= (\pi \otimes \pi \otimes \mathrm{Id}) \circ (\Delta \otimes \mathrm{Id}) \circ \delta_1(v) \\ &= (\delta_1 \otimes \mathrm{Id}) \circ \delta_1(v), \\ (\pi \otimes \pi \otimes \pi) \circ (\mathrm{Id} \otimes \Delta) \circ \Delta(v) &= \sum_{k=1}^{\infty} (\mathrm{Id} \otimes \pi \otimes \pi) \circ (\mathrm{Id} \otimes \Delta) \circ \delta_n(v) \\ &= (\mathrm{Id} \otimes \delta_1) \circ \delta_1(v) + (\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} + \mathrm{Id} \otimes \tau) \circ \delta_2(v). \end{aligned}$$

As a consequence, by the coassociativity of Δ,

$$(\delta_1 \otimes \mathrm{Id}) \circ \delta_1 - (\mathrm{Id} \otimes \delta_1) \circ \delta_1 = (\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} + \mathrm{Id} \otimes \tau) \circ \delta_2,$$

and it follows that,

$$(\delta_1 \otimes \mathrm{Id}) \circ \delta_1 - (\mathrm{Id} \otimes \delta_1) \circ \delta_1 = (\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} + \mathrm{Id} \otimes \tau) \circ ((\delta_1 \otimes \mathrm{Id}) \circ \delta_1 - \circ(\mathrm{Id} \otimes \delta_1) \circ \delta_1),$$

so $(\mathcal{V}, \delta_1)$ is a preLie coalgebra. □

In the case of interest here, namely that of $\mathcal{W}(G)$, define for $\omega \in \mathcal{W}(G)$,

$$\delta_n(\omega) := \sum_{c \in E_n \mathrm{AdC}(w)} \omega_c \otimes \omega^c,$$

with c an extended admissible cut *involving exactly n admissible cuts*, i.e.

$$c \in E_n\mathrm{AdC}(w) \iff \omega^c = \omega^{k_1,k'_1} \mid \cdots \mid \omega^{k_n,k'_n} \text{ where } \omega^{k_i,k'_i} \in \mathrm{AdC}(\omega).$$

By Theorem 8, the coproduct defined as in Eq. (7) with the above definition for the δ_n, namely Δ_{H}, is coassociative. Proposition 13 then implies that $(\mathcal{W}(G), \delta_1)$, $\delta_1 \equiv \Delta_{\mathrm{CP}}$, is a preLie coalgebra. In other terms, Theorem 6 may be seen as a corollary of Theorem 8.

5.2. Codendriform bialgebra

The brace co-structure on $\mathcal{W}(G)$ now implies that $\mathcal{T}(\mathcal{W}(G))$ is a codendriform bialgebra, a dual of the results of [11].

Denoting by $E\mathrm{AdC}_+(\omega) = E\mathrm{AdC}(\omega) \cup \{\mathbf{1}, \omega\}$ the set of extended admissible cuts of ω augmented by the empty cut and the total cut, recall that for any $n \geq 1$ walks $\omega_1, \cdots, \omega_n \in \mathcal{W}(G)$,

$$\Delta_{\mathrm{H}}(w_1 \mid \ldots \mid w_n) = \sum_{c_i \in EAdC_+(\omega_i)} (\omega_1)_{c_1} \mid \ldots \mid (\omega_n)_{c_n} \otimes \omega_1^{c_1} \mid \ldots \mid \omega_n^{c_n}.$$

Now define, for any nonempty word $\omega_1 \mid \ldots \mid \omega_n \in \mathcal{T}\langle \mathcal{W}(G) \rangle$, the maps

$$\Delta_{\prec}(\omega_1 \mid \ldots \mid \omega_n) := \sum_{\substack{c_i \in E\mathrm{AdC}_+(\omega_i),\\ (\omega_1)_{c_1} \neq \mathbf{1}}} (\omega_1)_{c_1} \mid \ldots \mid (\omega_n)_{c_n} \otimes \omega_1^{c_1} \mid \ldots \mid \omega_n^{c_n},$$

$$\Delta_{\succ}(w_1 \mid \ldots \mid w_n) := \sum_{\substack{c_i \in E\mathrm{AdC}_+(w_i),\\ (\omega_1)_{c_1} = \mathbf{1}}} (\omega_1)_{c_1} \mid \ldots \mid (\omega_n)_{c_n} \otimes \omega_1^{c_1} \mid \ldots \mid \omega_n^{c_n}.$$

Proposition 14. *$(\mathcal{T}(\mathcal{W}(G)), \Delta_{\prec}, \Delta_{\succ})$ is a codendriform bialgebra. Furthermore, for any $x \in \mathcal{T}(\mathcal{W}(G))$ with no constant term and any $y \in \mathcal{T}(\mathcal{W}(G))$, we have the two equalities $\Delta_{\prec}(x \mid y) = \Delta_{\prec}(x) \mid \Delta_{\mathrm{H}}(y)$, and $\Delta_{\succ}(x \mid y) = \Delta_{\succ}(x) \mid \Delta_{\mathrm{H}}(y)$.*

Proof. By the coassociativity of Δ_{H}, for any $\omega \in \mathcal{W}(G)$, $(\Delta_{\mathrm{H}} \otimes \mathrm{Id}) \circ \Delta_{\mathrm{H}}(\omega) = (\mathrm{Id} \otimes \Delta_{\mathrm{H}}) \circ \Delta_{\mathrm{H}}(\omega)$, that is

$$\sum_{c \in E\mathrm{AdC}_+(\omega)} \sum_{c' \in E\mathrm{AdC}_+(\omega_c)} (\omega_c)_{c'} \otimes (\omega_c)^{c'} \otimes \omega^c = \sum_{c \in E\mathrm{AdC}_+(\omega)} \sum_{c' \in E\mathrm{AdC}_+(\omega^c)} \omega_c \otimes (\omega^c)_{c'} \otimes (\omega^c)^{c'}.$$

Thus, there exists a set $E\mathrm{AdC}_+^{(2)}(\omega)$ such that the above may be put in the form

$$\sum_{c \in E\mathrm{AdC}_+^{(2)}(\omega)} \omega_c \otimes \omega^{c(1)} \otimes \omega^{c(2)}.$$

Then, using this notation,

$$\begin{aligned}(\Delta_{\mathrm{H}} \otimes \mathrm{Id}) \circ \Delta_{\succ}(\omega_1 | \ldots | \omega_n) &= (\mathrm{Id} \otimes \Delta_{\succ}) \circ \Delta_{\succ}(\omega_1 | \ldots | \omega_n)\\ &= \sum_{\substack{c_i \in EAdC_+^{(2)}(w_i),\\ (\omega_1)_{c_1} = \omega_1^{c_1(1)} = \mathbf{1}}} (\omega_1)_{c_1} \mid \ldots \mid (\omega_n)_{c_1} \otimes \omega_1^{c_1(1)} \mid \ldots \mid \omega_n^{c_n(1)} \otimes \omega_1^{c_1(2)} \mid \ldots \mid \omega_n^{c_n(2)},\end{aligned}$$

$$(\Delta_{\succ}\otimes \mathrm{Id})\circ\Delta_{\prec}(\omega_1|\dots|\omega_n) = (\mathrm{Id}\otimes\Delta_{\prec})\circ\Delta_{\succ}(\omega_1|\dots|\omega_n)$$
$$= \sum_{\substack{c_i\in E\mathrm{AdC}_+^{(2)}(w_i),\\ (\omega_1)_{c_1}=\mathbf{1},\,\omega_1^{c_1(1)}\neq\mathbf{1}}} (\omega_1)_{c_1} \mid \dots \mid (\omega_n)_{c_1} \otimes \omega_1^{c_1(1)} \mid \dots \mid \omega_n^{c_n(1)} \otimes \omega_1^{c_1(2)} \mid \dots \mid \omega_n^{c_n(2)},$$
$$(\Delta_{\prec}\otimes \mathrm{Id})\circ\Delta_{\prec}(\omega_1|\dots|\omega_n) = (\mathrm{Id}\otimes\Delta_{\mathrm{H}})\circ\Delta_{\prec}(\omega_1|\dots|\omega_n)$$
$$= \sum_{\substack{c_i\in E\mathrm{AdC}_+^{(2)}(w_i),\\ (\omega_1)_{c_1}\neq\mathbf{1}}} (\omega_1)_{c_1} \mid \dots \mid (\omega_n)_{c_1} \otimes \omega_1^{c_1(1)} \mid \dots \mid \omega_n^{c_n(1)} \otimes \omega_1^{c_1(2)} \mid \dots \mid \omega_n^{c_n(2)}.$$

□

6. Cacti, ladders and corollas

Recall from Definition 6 that a cactus is a kind of "disentangled" walk resembling a self-avoiding skeleton on which bouquets of ladders are attached. Given that by the proof of Theorem 4 all walks are chronologically equivalent to cacti, it seems intuitive that bouquets and ladders are basic building blocks of walks and ought to be associated to sub-algebras of the walk algebras. In this section we formalize this observation by showing first that cacti, ladders and corollas (a special type of bouquets) give rise to sub-Hopf algebras of the tensor and symmetric algebras of all walks; and secondly that the mapping from walks to cacti effected by the map C defined in the proof of Theorem 4 generates Hopf algebra morphisms.

In a forthcoming work, using the permutative non-associative product nesting and the NAP-copreLie structure it forms with Δ_{CP}, we will formalize and exploit algebraically the construction of walks from bouquets and ladders based on Lawler's process.

6.1. Hopf subalgebras associated to cacti, ladders and corollas

Definition 14 (Ladder). Let G be a digraph. A ladder with root r_1 and of height $n\in\mathbb{N}\backslash\{0\}$ is a closed walk made of a collection $\mathrm{Cycl}_1, \dots, \mathrm{Cycl}_n$ of *simple cycles* with roots $r_1, \dots, r_n$, respectively, and such that:

i) $V(\mathrm{Cycl}_k)\cap V(\mathrm{Cycl}_{k+1}) = \{r_{k+1}\}$ for any $k\in\{1,\dots,n-1\}$,

ii) $V(\mathrm{Cycl}_k)\cap V(\mathrm{Cycl}_l) = \emptyset$ whenever $|k-l|>1$.

The vector space spanned by the ladders of G is denoted by $\mathrm{Lad}(G)$. The space $\mathcal{T}\langle\mathrm{Lad}(G)\rangle$ (respectively $\mathcal{S}\langle\mathrm{Lad}(G)\rangle$) is the tensor algebra (respectively the symmetric algebra) generated by $\mathrm{Lad}(G)$.

Definition 15 (Corolla). Let G be a digraph. A corolla with root r in G is a closed walk made of $n\in\mathbb{N}\backslash\{0\}$ simple cycles $\mathrm{Cycl}_1, \dots, \mathrm{Cycl}_n$, all with a common root r. Corollas are bouquets of simple cycles.

The vector space spanned by all corollas (respectively corollas of root r) of G is $\mathrm{Cor}(G)$ (respectively $\mathrm{Cor}_r(G)$). The space $\mathcal{T}\langle\mathrm{Cor}(G)\rangle$ (respectively $\mathcal{S}\langle\mathrm{Cor}(G)\rangle$) is the tensor algebra (respectively the symmetric algebra) generated by $\mathrm{Cor}(G)$. We define the spaces $\mathcal{T}\langle\mathrm{Cor}_r(G)\rangle$ and $\mathcal{S}\langle\mathrm{Cor}_r(G)\rangle$ similarly from $\mathrm{Cor}_r(G)$.

Example 16. The walk $123454321\in\mathrm{Lad}(G)$ is a ladder, while walks $111\in\mathrm{Cor}_1(G)$ and $123412451\in\mathrm{Cor}_1(G)$ are corollas with root 1.

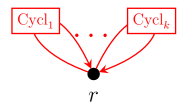
Proposition 15. *Let G be a digraph and $r\in V(G)$. Then,*

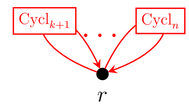
1. $(\mathcal{T}\langle\mathrm{Lad}(G)\rangle,\bullet,\Delta_{\mathrm{H}})$, $(\mathcal{T}\langle\mathrm{Cor}_r(G)\rangle,\bullet,\Delta_{\mathrm{H}})$, $(\mathcal{T}\langle\mathrm{Cor}(G)\rangle,\bullet,\Delta_{\mathrm{H}})$ *and* $(\mathcal{T}\langle\mathrm{Cact}(G)\rangle,\bullet,\Delta_{\mathrm{H}})$ *are Hopf subalgebras of* $(\mathcal{T}\langle\mathcal{W}(G)\rangle,\bullet,\Delta_{\mathrm{H}})$.

2. $(\mathcal{S}\langle \mathrm{Lad}(G)\rangle, \square, \Delta_{\mathrm{H}})$, $(\mathcal{S}\langle \mathrm{Cor}_r(G)\rangle, \square, \Delta_{\mathrm{H}})$, $(\mathcal{S}\langle \mathrm{Cor}(G)\rangle, \square, \Delta_{\mathrm{H}})$ *and* $(\mathcal{S}\langle \mathrm{Cact}(G)\rangle, \square, \Delta_{\mathrm{H}})$ *are Hopf subalgebras of* $(\mathcal{S}\langle \mathcal{W}(G)\rangle, \square, \Delta_{\mathrm{H}})$.

Proof. Firstly, the claims regarding $(\mathcal{T}\langle \mathrm{Lad}(G)\rangle, \bullet, \Delta_{\mathrm{H}})$ and $(\mathcal{S}\langle \mathrm{Lad}(G)\rangle, \square, \Delta_{\mathrm{H}})$ are shown by direct calculation.

Secondly, let ω be a corolla with root $r \in V(G)$ comprising $n \in \mathbb{N}\backslash\{0\}$ simple cycles $\mathrm{Cycl}_{1\leq k\leq n}$. Let $v \in V(G)$ be a vertex other than the root r visited by ω. Since $\mathrm{Cycl}_1, \ldots, \mathrm{Cycl}_n$ are simple cycles, if v is visited several times by ω then two instances of v cannot be found within a unique simple cycle. But by using Remark 4 equivalent to Definition 4 for the loop- erased sections, any subwalk $\omega^{l,l'} = w_l \cdots w_{l'}$ with $w_l = w_{l'} = v$ is not an admissible cut of ω, as it is not a valid loop-erased section of ω. Then all the admissible cuts of ω take place at the root r,

$$\Delta_{\mathrm{H}}(\omega) = \omega \otimes 1 + 1 \otimes \omega + \sum_{p=1}^{n-1}$$


$$\otimes$$


which implies the claims for $(\mathcal{T}\langle \mathrm{Cor}(G)\rangle, \bullet, \Delta_{\mathrm{H}})$, $(\mathcal{S}\langle \mathrm{Cor}_i(G)\rangle, \square, \Delta_{\mathrm{H}})$ and $(\mathcal{S}\langle \mathrm{Cor}(G)\rangle, \square, \Delta_{\mathrm{H}})$.

Thirdly, the claims about $(\mathcal{T}\langle \mathrm{Cact}(G)\rangle, \bullet, \Delta_{\mathrm{H}})$ and $(\mathcal{S}\langle \mathrm{Cact}(G)\rangle, \square, \Delta_{\mathrm{H}})$ both follow from Proposition 7 and the fact that an admissible cut of a walk ω is, by definition, a loop-erased section of ω. $\square$

Remark 9. Since Proposition 15 establishes Hopf algebra structures on the tensor algebras generated by ladders, corollas and cacti, the constructions of §5 extend to these walks as well. That is, there are brace coalgebras and codendriform bialgebras on ladders, corollas and cacti and these are sub coalgebras of the structures of §5 on all walks.

6.2. The cactus map generates Hopf algebra morphisms

We now show that the map C defined in Eq. (1) which sends a walk ω to a cactus generates Hopf algebra morphisms. Recall that, by definition, $C(\omega)$ is a cactus in the complete graph $K_{\mathbb{N}}$ with $V(K_{\mathbb{N}}) = \mathbb{N}$.

Let $\mathcal{I}_{\mathbb{N}}$ be the set of the injective maps $\mathbb{N} \to \mathbb{N}$. For $f \in \mathcal{I}_{\mathbb{N}}$ and $\omega = w_0 \cdots w_\ell \in \mathcal{W}(K_{\mathbb{N}})$, we denote by $f(\omega) \in \mathcal{W}(K_{\mathbb{N}})$ the walk defined by $f(\omega) := f(w_0) \ldots f(w_\ell)$.

Definition 16. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the vector spaces defined by:

$$\mathcal{J}_1 := \mathrm{Span}\big(\omega_1 \,|\, \ldots \,|\, \omega_n - f_1(\omega_1) \,|\, \ldots \,|\, f_n(\omega_n);\ n \in \mathbb{N}\backslash\{0\}, \omega_i \in \mathrm{Cact}(K_{\mathbb{N}}), f_i \in \mathcal{I}_{\mathbb{N}}\big),$$
$$\mathcal{J}_2 := \mathrm{Span}\big(\omega_1 \,\square\, \ldots \,\square\, \omega_n - f_1(\omega_1) \,\square\, \ldots \,\square\, f_n(\omega_n);\ n \in \mathbb{N}\backslash\{0\}, \omega_i \in \mathrm{Cact}(K_{\mathbb{N}}), f_i \in \mathcal{I}_{\mathbb{N}}\big).$$

Proposition 16. *The vector space* $\mathcal{J}_1$ *(respectively* $\mathcal{J}_2$*) is a Hopf biideal of* $\mathcal{T}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle$ *(respectively* $\mathcal{S}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle$*).*

Proof. We prove the result for $\mathcal{J}_1$. The reasoning for $\mathcal{J}_2$ is entirely similar.

Let $\omega = w_0 \cdots w_\ell \in \mathcal{W}(G)$, then for any injective map $f \in \mathcal{I}_{\mathbb{N}}$, the length of $f(\omega)$ is still ℓ and we have the relation Eq. (2), that is

$$\omega^{k,k'} \in \mathrm{AdC}(\omega) \iff f(\omega)^{k,k'} \in \mathrm{AdC}(f(\omega)). \tag{8}$$

Therefore, if $\omega \in \mathrm{Cact}(G)$, $f(\omega)$ is also a cactus.

Now let $\alpha := \omega_1|\ldots|\omega_n - f_1(\omega_1)|\ldots|f_n(\omega_n)$ be a generator of $\mathcal{J}_1$ and $\beta := \tau_1|\ldots|\tau_m \in \mathcal{T}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle$,

$$\begin{aligned}\alpha \bullet \beta &= \omega_1 \,|\, \ldots \,|\, \omega_n \,|\, \tau_1 \,|\, \ldots \,|\, \tau_m - f_1(\omega_1) \,|\, \ldots \,|\, f_n(\omega_n) \,|\, \tau_1 \,|\, \ldots \,|\, \tau_m \\ &= \omega_1 \,|\, \ldots \,|\, \omega_n \,|\, \tau_1 \,|\, \ldots \,|\, \tau_m - f_1(\omega_1) \,|\, \ldots \,|\, f_n(\omega_n) \,|\, \mathrm{Id}(\tau_1) \,|\, \ldots \,|\, \mathrm{Id}(\tau_m).\end{aligned}$$

So we obtain $\alpha \bullet \beta \in \mathcal{J}_1$ and similarly $\beta \bullet \alpha \in \mathcal{J}_1$. As a consequence, $\mathcal{J}$ is an ideal.

Let $\omega = w_0 \dots w_\ell \in \mathrm{Cact}(G)$ and $f \in \mathcal{I}_{\mathbb{N}}$. By injectivity of f for $c \in E\mathrm{AdC}(\omega)$ with $\omega^c := \omega^{k_1,k_1';\dots;k_nk_n'}$, we have

$$f(\omega)_c = f(\omega)_{k_1,k_1';\dots;k_nk_n'} = f(\omega_{k_1,k_1';\dots;k_nk_n'}) = f(\omega_c).$$

Therefore

$$\begin{aligned}\Delta_{\mathrm{H}}(\omega - f(\omega)) &= (\omega - f(\omega)) \otimes \mathbf{1} + \mathbf{1} \otimes (\omega - f(\omega)) + \sum_{c \in E\mathrm{AdC}(\omega)} \big\{\omega_c \otimes \omega^c - f(\omega)_c \otimes f(\omega)^c\big\},\\ &= (\omega - f(\omega)) \otimes \mathbf{1} + \mathbf{1} \otimes (\omega - f(\omega)) + \sum_{c \in E\mathrm{AdC}(\omega)} \big\{\omega_c \otimes \omega^c - \omega_c \otimes f(\omega)^c\big\}\\ &\quad + \sum_{c \in E\mathrm{AdC}(\omega)} \big\{\omega_c \otimes f(\omega)^c - f(\omega_c) \otimes f(\omega)^c\big\}.\end{aligned}$$

This shows that $\Delta_{\mathrm{H}}(\omega - f(\omega)) \in \mathcal{T}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle \otimes \mathcal{J}_1 + \mathcal{J}_1 \otimes \mathcal{T}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle$. Since furthermore Δ_{H} is an algebra morphism, we conclude that $\mathcal{J}_1$ is a coideal.

Finally, by Eq. (2), Theorem 11 and the fact the antipode is an algebra antimorphism, we get $S(\mathcal{J}_1) \subset \mathcal{J}_1$. □

Remark 10. The elements of $\mathcal{T}\langle \mathrm{Cact}\rangle(K_{\mathbb{N}})/\mathcal{J}_1$ and $\mathcal{S}\langle \mathrm{Cact}\rangle(K_{\mathbb{N}})/\mathcal{J}_2$ can be seen as cacti where the node labels have been forgotten since the node labels are defined modulo the action of $\mathcal{I}_{\mathbb{N}}$. These Hopf algebras can thus legitimately be called the tensor and symmetric Hopf algebras of unlabeled cacti, respectively.

By direct calculation,

Proposition 17. *The degree map* deg *makes* $\mathcal{T}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle/\mathcal{J}_1$ *and* $\mathcal{S}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle/\mathcal{J}_2$ *into graded Hopf algebras.*

Theorem 18. *Let G be a digraph. Let* $\Phi_1 : \mathcal{T}\langle \mathcal{W}(G)\rangle \to \mathcal{T}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle/\mathcal{J}_1$ *and* $\Phi_2 : \mathcal{S}\langle \mathcal{W}(G)\rangle \to \mathcal{T}\langle \mathrm{Cact}(K_{\mathbb{N}})\rangle/\mathcal{J}_2$ *be the two algebra morphisms such that $\Phi_i(\omega)$ is the unlabeled cactus obtained from $C(\omega)$ by forgetting all its node labels. Then Φ_1 and Φ_2 are Hopf algebra morphisms.*

Proof. By definition, the cardinalities of $V(\omega)$ and $V(C(\omega))$ are equal, $C(\omega)$ is a cactus and Eq. (2) holds. By the definition of Δ_{H} and the formulas of the antipode given in Theorem 11 and Corollary 12, we prove the theorem. □

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